Available online at www.sciencedirect.com

SCIENCE DIRECT®

International Journal of Solids and Structures 43 (2006) 3057–3086

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

A family of metric strains and conjugate stresses, prolonging usual material laws from small to large transformations

A. Curnier [a], Ph. Zysset [b,*]

[a] *Laboratoire de Mécanique Appliquée et d'Analyse de Fiabilité EPFL CH-1015 Lausanne, Switzerland*
[b] *Institut für Leichtbau und Struktur-Biomechanik TU-Wien Gußhausstrasse 27-29 A-1040 Wien, Austria*

Received 4 February 2005; received in revised form 3 June 2005
Available online 9 August 2005

Abstract

A new family of simple generalized strains and conjugate stresses based on the material metric (right Cauchy–Green) tensor is proposed. It includes an interesting quasilinear pair. It is a close approximation of the Seth–Hill family, with the advantage of being easier to calculate. It extends the realm of application of the classical theories of linear elasticity and perfect plasticity from small to large transformations for isotropic and anisotropic materials without any modification.

Keywords: Strain; Stress; Generalized strain; Conjugate stress; Nonlinear elasticity; Large transformations; Rank-one convexity

1. Introduction

1.1. Motivation

In nonlinear mechanics, the form assumed by the stress–strain law of a material depends on the stress–strain pair selected for formulating it. A major trend is to use the simplest strain–stress pair namely the Green–St.Venant quadratic strain and conjugate second Piola–Kirchhoff stress and to defer all the complexity of the material response to the stress–strain law, which is a sound approach. A less used but appealing alternative consists in resorting to a more elaborate strain–stress pair such as the Cauchy–Biot linear

* Corresponding author. Tel.: +43 1 58801 31723; fax: +43 1 58801 31799.
E-mail address: philippe.zysset@ilsb.tuwien.ac.at (Ph. Zysset).

0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2005.06.015

strain and conjugate Biot–Ziegler stress, which significantly simplifies the stress–strain law eventually down to its small transformation expression. Of course, simplifying the modelling of materials by complicating the description of strain and stress may appear vacuous to many mechanicians. However, the prospect of extending the realm of an-isotropic linear elasticity or perfect plasticity from small to large transformations without any modification from their parts represents a sufficient incentive to the other rheologists.

1.2. Precedents

Several pairs of conjugate strain–stress measures have long been identified in nonlinear mechanics, for which extended bibliographies can be found in (Curnier and Rakotomanana, 1991; Guan-Suo et al., 1999). Among the material pairs, the *Green quadratic strain* with the *Kirchhoff stress* for conjugate and the *Karni quadhyperbolic strain* with the *Rivlin stress* for conjugate play a fundamental role (Truesdell, 1952; Green and Rivlin, 1964; Karni and Reiner, 1964; Hill, 1968). They are both formulated in terms of the metric but they are called according to their degree in the stretch. The quadratic pair uses the original form for reference, whereas the quadhyperbolic pair uses the actual form for the same purpose. These two "inverse" pairs play a reference role not only because they are simple but also because they can be viewed as an upper and lower bound for other candidate pairs which must hence lie in between.

The best known family of generalized strains with conjugate stresses, along this interpolating idea, is the *Seth–Hill stretch family* (Doyle and Ericksen, 1956; Seth, 1964; Hill, 1968). The main shortcoming of this family is that except for the quadratic and quadhyperbolic cases, the stretch strains are difficult to compute and their conjugate stresses are even harder to calculate.

1.3. Proposition

In this article, a new family of simple material strains called *metric strains* is proposed in the form of a *convex combination* of the quadratic and quadhyperbolic strains, together with their *conjugate material stresses*. The *metric* family is parametrized by a real number n and therefore contains infinitely many members. For integer values of n, it includes the Green–Kirchhoff and Karni–Rivlin pairs already introduced above for $n = \pm 2$, plus three new intermediate members called the *quasilinear* ($n = 1$), *quasilogarithmic* ($n = 0$) and *quasihyperbolic* ($n = -1$) pairs because they are simple approximations of the corresponding linear, logarithmic and hyperbolic pairs of the stretch family. It will be shown that the metric family is a close approximation of the stretch family, while being easier to calculate. The metric stress–strain family opens the way for formulating gradual families of constitutive laws at large transformations. For instance, postulating a *linear* elastic metric law produces at once a family of nonlinear elastic nominal laws. A thorough analysis of the rank-one convexity properties of the isotropic elastic energy density from which the metric law derives, shows that, within the range $0 \leqslant n \leqslant 1$, the metric law is rank-one monotone (i.e. its energy density is rank-one convex) over a much wider region around the origin than the classical StVenant–Kirchhoff law ($n = 2$) and the opposite law ($n = -2$), which are not even monotone in compression and in tension, respectively. This is a major improvement from a fundamental standpoint since rank-one monotony is a necessary condition for the existence of a solution to the corresponding boundary value problem. In fact, similar trends have been already observed by Hill (1978), Bruhns et al. (2001) using the Seth–Hill family, where $n = 0$ stands for the logarithmic strain.

1.4. Outline

The concepts of metric strain and conjugate stress are presented in Sections 2 and 3, respectively. "Metric" elasticity is then formulated in Section 4 and its rank-one convexity properties are analyzed for the isotropic case in Section 5. The homogeneous stress–strain states of tension–dilatation, traction–elongation and shear–glide are then illustrated in Section 6.

1.5. Notation

Hereafter, scalars are denoted by italic letters (e.g. t), vectors by bold face minuscules ($\mathbf{x}$), second-order tensors by bold face majuscules ($\mathbf{F}$) and fourth-order tensors by outline majuscules ($\mathbb{I}$).

With these notations, let $\mathbf{F}$ denote the *Euler nominal strain* defined as the gradient $\mathbf{F} = \nabla_{\mathbf{x}}\mathbf{y}$ of the motion $\mathbf{y} : (\mathbf{x}, t) \mapsto \mathbf{y}(\mathbf{x}, t)$ and $\mathbf{P}$ the conjugate *Piola nominal stress* defined through the nominal version of Cauchy's theorem $\mathbf{p} = \mathbf{Pn}$, where $\mathbf{p}$ is the corresponding traction vector and $\mathbf{n}$ the original normal. $\mathbf{P}$ and $\mathbf{F}$ are *conjugate* or *dual* because their internal power is equal to the external power supplied to the solid: $\int_\Omega \mathbf{P} : \dot{\mathbf{F}}\,\mathrm{d}V = \int_{\partial\Omega} \mathbf{p} \cdot \dot{\mathbf{y}}\,\mathrm{d}A$ (where Ω is the solid original form and $\dot{\mathbf{y}}$ the particle velocity). However $\mathbf{F}$ and $\mathbf{P}$ are not symmetric and not objective, which complicates the direct formulation of constitutive laws in their terms.

The right *Cauchy* (symmetric, positive-definite) *material metric* $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ (hereafter referred to as the metric tensor) and the *Kirchhoff* (symmetric) *material stress* $\mathbf{S}$ defined through $\mathbf{P} = \mathbf{FS}$ are more appropriate for this purpose. $\mathbf{S}$ and $\mathbf{C}/2$ are also conjugate since they develop the same power as $\mathbf{P}$ and $\mathbf{F}$, i.e. $\mathbf{S} : \dot{\mathbf{C}}/2 = \mathbf{P} : \dot{\mathbf{F}}$. So-called *material* strain–stress pairs, formulated in terms of $\mathbf{C}$ and $\mathbf{S}$ can simplify even further the task of formulating material laws. The right Cauchy *material stretch* $\mathbf{U} = \sqrt{\mathbf{C}}$, occuring in the right polar decomposition $\mathbf{F} = \mathbf{RU}$, will also serve as a standard of comparison. The reader is referred to e.g. (Truesdell and Toupin, 1960; Truesdell and Noll, 1965; Eringen, 1975; Chadwick, 1976; Hill, 1978; Gurtin, 1981; Curnier, 2004) for complete treatments. Finally the following tensor products (cf. e.g. Curnier, 1994; Curnier, 2004) will be used:

- *vector dyadic* $\quad [\mathbf{a} \otimes \mathbf{b}]\mathbf{x} = (\mathbf{x} \cdot \mathbf{b})\mathbf{a},\ \forall \mathbf{x} \quad [\mathbf{e}_i \otimes \mathbf{e}_i]\mathbf{x} = x_i\mathbf{e}_i$
- *tensor dyadic* $\quad [\mathbf{A} \otimes \mathbf{B}]\mathbf{X} = (\mathbf{X} : \mathbf{B})\mathbf{A},\ \forall \mathbf{X} \quad [\mathbf{I} \otimes \mathbf{I}]\mathbf{X} = (\operatorname{tr} \mathbf{X})\mathbf{I}$
- *tensor product* $\quad [\mathbf{A} \underline{\otimes} \mathbf{B}]\mathbf{X} = \mathbf{AXB}^{\mathrm{T}},\ \forall \mathbf{X} \quad [\mathbf{I} \underline{\otimes} \mathbf{I}]\mathbf{X} = \mathbf{X}$
- *transp. product* $\quad [\mathbf{A} \overline{\otimes} \mathbf{B}]\mathbf{X} = \mathbf{AX}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}},\ \forall \mathbf{X} \quad [\mathbf{I} \overline{\otimes} \mathbf{I}]\mathbf{X} = \mathbf{X}^{\mathrm{T}}$
- *symm. product* $\quad [\mathbf{A} \overline{\underline{\otimes}} \mathbf{B}]\mathbf{X} = \mathbf{AXB}^{\mathrm{T}},\ \forall \mathbf{X} = \mathbf{X}^{\mathrm{T}} \quad [\mathbf{I} \overline{\underline{\otimes}} \mathbf{I}]\mathbf{X} = \mathbf{X}$

hence $\mathbf{A} \overline{\underline{\otimes}} \mathbf{B} = \frac{1}{2}(\mathbf{A} \underline{\otimes} \mathbf{B} + \mathbf{A} \overline{\otimes} \mathbf{B})$. The summation convention on repeated indexes is also used.

2. Metric strain

The concept of generalized strain was introduced by Doyle and Ericksen (1956) and by Seth (1964), generalized by Hill (1968), Hill (1978) and studied by Ogden (1974), Ogden (1984) and Curnier and Rakotomanana (1984), Curnier and Rakotomanana (1991), among many others.

2.1. Metric provision

A claimed objective of this study is to propose *simple, easy-to-calculate*, strain measures. Comparing the available "easy" Green and Karni strains, to the "harder" intermediate members of the stretch family, it is clear that a dependence on the *metric* $\mathbf{C}$ must be preferred to a dependence on the stretch $\mathbf{U}$, in order to avoid extraction of the square root $\mathbf{U} = \sqrt{\mathbf{C}}$ and, later on, the derivation of the stretch rate $\dot{\mathbf{U}}$. Throughout this article, the term metric tensor will be used as a shorthand for right Cauchy–Green tensor. In short, computational facility suggests working with metric integer powers rather than with stretch ones.

2.2. Generalized strain

A *generalized material strain* $\mathbf{G}$ is defined as a symmetric tensor valued smooth monotone function of the symmetric positive-definite material metric tensor $\mathbf{C}$, which derives from a convex potential, vanishes in the original form Ω (where $\mathbf{C} = \mathbf{I}$) and has a half-unit gradient there

$$\mathbf{G} : \mathbf{C} \mapsto \mathbf{G}(\mathbf{C}) = \mathbf{G}^{\mathrm{T}}(\mathbf{C}), \quad \mathbf{G}(\mathbf{I}) = \mathbf{O} \tag{1}$$

$$\nabla_{\mathbf{C}}\mathbf{G}(\mathbf{C}) = \nabla^{\mathrm{T}}_{\mathbf{C}}\mathbf{G}(\mathbf{C}) \succ 0, \quad \nabla_{\mathbf{C}}\mathbf{G}(\mathbf{I}) = \underline{\overline{\mathbb{I}}}/2 \tag{2}$$

In these relations, $\nabla_{\mathbf{C}}\mathbf{G}$ denotes the (fourth-order) gradient of $\mathbf{G}$ with respect to $\mathbf{C}$ with major symmetry $\nabla^{\mathrm{T}}_{\mathbf{C}}\mathbf{G} = \nabla_{\mathbf{C}}\mathbf{G} \iff \mathbf{X} : \nabla_{\mathbf{C}}\mathbf{G}\mathbf{Y} = \mathbf{Y} : \nabla_{\mathbf{C}}\mathbf{G}\mathbf{X},\ \forall \mathbf{X}, \mathbf{Y}$ (in addition to the two minor ones) and positive-definiteness $\succ 0 \iff \mathbf{X} : \nabla_{\mathbf{C}}\mathbf{G}\mathbf{X} > 0,\ \forall \mathbf{X} = \mathbf{X}^{\mathrm{T}} \neq \mathbf{O}$; $\underline{\overline{\mathbb{I}}} \equiv \mathbf{I} \,\underline{\overline{\otimes}}\, \mathbf{I}$ denotes the (fourth-order) identity tensor for symmetric (second-order) tensors, i.e. $\underline{\overline{\mathbb{I}}}\mathbf{X} = \mathbf{X},\ \forall \mathbf{X} = \mathbf{X}^{\mathrm{T}}$ (cf. Curnier, 2004).

Besides computational ease, dependence on $\mathbf{C}$ guarantees *objectivity*. Smoothness and monotony guarantee *bijectivity* between $\mathbf{C}$ and $\mathbf{G}(\mathbf{C})$ and hereby existence of a *smooth inverse* $\mathbf{G}^{-1}$. Existence of a strain potential guarantees *strain path indifference*. The two *consistency* conditions imply that the strain vanishes in the original form and coincides with the ‘‘small’’ Cauchy strain about it. All classical strains are generalized strains.

Besides the minimal requirements (1) and (2), a generalized strain $\mathbf{G}$ should be a *coercive* function over its domain of definition $\mathbb{S}\mathrm{ym}_+$ (the convex cone of positive definite symmetric tensors), meaning that its norm should tend to infinity on its boundary $\partial\mathbb{S}\mathrm{ym}_+$, i.e., $\|\mathbf{G}(\mathbf{C})\| \to \infty$ as $\|\mathbf{C}\| \to 0$ and $\|\mathbf{C}\| \to \infty$, where $\|\mathbf{X}\| = \sqrt{\mathrm{tr}(\mathbf{X}^{\mathrm{T}}\mathbf{X})}$. Coerciveness means that a strain should tend to plus infinity when a bar is elongated to infinity and to minus infinity when it is shortened to zero. It is kept as a desirable attribute rather than a requirement however, because the Green and Karni strains violate it in compression and in tension, respectively.

2.3. Isotropic strain

In addition to (1) and (2), a generalized strain $\mathbf{G}$ is usually required to be an *isotropic* function of $\mathbf{C}$ for excluding an artificial geometric anisotropy which would interfere with an eventual genuine material anisotropy

$$\mathbf{G}(\mathbf{R}\mathbf{C}\mathbf{R}^{\mathrm{T}}) = \mathbf{R}\mathbf{G}(\mathbf{C})\mathbf{R}^{\mathrm{T}} \quad \forall \mathbf{R} = \mathbf{R}^{-\mathrm{T}} \tag{3}$$

In short, isotropy infers *material direction indifference* of the strain measure. The theorem of representation of isotropic functions of a symmetric tensor provides the general form of an isotropic strain as a nonlinear combination of three consecutive powers of $\mathbf{C}$ called ‘‘generators’’, with coefficients depending on three independent invariants of $\mathbf{C}$. In view of the expressions of the classical Green and Karni strains, these three generators are preferably taken to be $\mathbf{C}$, $\mathbf{I}$ and $\mathbf{C}^{-1}$ and then the invariants to be their primitives for simplicity

$$\begin{aligned}
&\Gamma_k (k = 1, 0, -1), \quad && \Gamma_1 = \mathrm{tr}\,\mathbf{C}^2/2, \quad && \Gamma_0 = \mathrm{tr}\,\mathbf{C}, \quad && \Gamma_{-1} = \ln(\det \mathbf{C}) = \mathrm{tr}\,(\ln \mathbf{C})\\
&\nabla_{\mathbf{C}}\Gamma_k = \mathbf{C}^k, && \nabla_{\mathbf{C}}\Gamma_1 = \mathbf{C}, && \nabla_{\mathbf{C}}\Gamma_0 = \mathbf{I}, && \nabla_{\mathbf{C}}\Gamma_{-1} = \mathbf{C}^{-1}\\
&\nabla^2_{\mathbf{C}}\Gamma_k = \nabla_{\mathbf{C}}\mathbf{C}^k, && \nabla^2_{\mathbf{C}}\Gamma_1 = \mathbf{I} \,\underline{\overline{\otimes}}\, \mathbf{I}, && \nabla^2_{\mathbf{C}}\Gamma_0 = \mathbb{O}, && \nabla^2_{\mathbf{C}}\Gamma_{-1} = -\mathbf{C}^{-1} \,\underline{\overline{\otimes}}\, \mathbf{C}^{-1}
\end{aligned}$$

where the gradients of the metric determinant and inverse are recalled to be $\nabla_{\mathbf{C}} \det \mathbf{C} = \det \mathbf{C}\ \mathbf{C}^{-1}$ and $\nabla_{\mathbf{C}}\mathbf{C}^{-1} = -\mathbf{C}^{-1} \,\underline{\overline{\otimes}}\, \mathbf{C}^{-1}$, respectively.

Hence, an *isotropic material strain* is defined as a smooth monotone isotropic function based on the three generators $\mathbf{C}$, $\mathbf{I}$ and $\mathbf{C}^{-1}$, which vanishes in the original form Ω and has a half-unit gradient there

$$\begin{aligned}
\mathbf{G}(\mathbf{C}) &= G_i(\Gamma_1,\Gamma_0,\Gamma_{-1})\mathbf{C}^i \\
&= G_1(\Gamma_1,\Gamma_0,\Gamma_{-1})\mathbf{C} + G_0(\Gamma_1,\Gamma_0,\Gamma_{-1})\mathbf{I} + G_{-1}(\Gamma_1,\Gamma_0,\Gamma_{-1})\mathbf{C}^{-1} \\
G_1(3/2,3,0) &+ G_0(3/2,3,0) + G_{-1}(3/2,3,0) = 0
\end{aligned} \tag{4}$$

$$\begin{aligned}
\nabla_{\mathbf{C}}\mathbf{G}(\mathbf{C}) &= \frac{\partial G_i}{\partial \Gamma_j}(\Gamma_1,\Gamma_0,\Gamma_{-1})\mathbf{C}^i \otimes \mathbf{C}^j + G_k(\Gamma_1,\Gamma_0,\Gamma_{-1})\nabla_{\mathbf{C}}\mathbf{C}^k \\
&= \frac{\partial G_1}{\partial \Gamma_1}\mathbf{C}\otimes\mathbf{C} + \frac{\partial G_1}{\partial \Gamma_0}\mathbf{C}\otimes\mathbf{I} + \frac{\partial G_1}{\partial \Gamma_{-1}}\mathbf{C}\otimes\mathbf{C}^{-1} \\
&\quad + \frac{\partial G_0}{\partial \Gamma_1}\mathbf{I}\otimes\mathbf{C} + \frac{\partial G_0}{\partial \Gamma_0}\mathbf{I}\otimes\mathbf{I} + \frac{\partial G_0}{\partial \Gamma_{-1}}\mathbf{I}\otimes\mathbf{C}^{-1} \\
&\quad + \frac{\partial G_{-1}}{\partial \Gamma_1}\mathbf{C}^{-1}\otimes\mathbf{C} + \frac{\partial G_{-1}}{\partial \Gamma_0}\mathbf{C}^{-1}\otimes\mathbf{I} + \frac{\partial G_{-1}}{\partial \Gamma_{-1}}\mathbf{C}^{-1}\otimes\mathbf{C}^{-1} \\
&\quad + G_1\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I} - G_{-1}\mathbf{C}^{-1}\,\overline{\underline{\otimes}}\,\mathbf{C}^{-1} \succ 0 \\
\Sigma_{i,j}\frac{\partial G_i}{\partial \Gamma_j}&(3/2,3,0) = 0,\ G_1(3/2,3,0) - G_{-1}(3/2,3,0) = 1/2
\end{aligned} \tag{5}$$

Here, the G_i are 3 scalar functions of the $3\Gamma_j$ and their gradients are calculated by the chain rule $\nabla_{\mathbf{C}}G_i = (\partial G_i/\partial\Gamma_j)\nabla_{\mathbf{C}}\Gamma_j = (\partial G_i/\partial\Gamma_j)\mathbf{C}^j$ $(i,j = 1,0,-1)$ (cf. Curnier, 2004) and evaluated at $(\Gamma_1,\Gamma_0,\Gamma_{-1})$. The major symmetry (2) translates into that of the coefficient Jacobian matrix: $\partial G_i/\partial\Gamma_j = \partial G_j/\partial\Gamma_i$.

Using the spectral decomposition of the metric tensor

$$\mathbf{C} = \gamma_a \mathbf{c}_a \otimes \mathbf{c}_a \quad (a = 1,3; 0 < \gamma_a < \infty; \mathbf{c}_a \cdot \mathbf{c}_b = \delta_{ab}) \tag{6}$$

where γ_a are the 3 positive real eigenvalues of $\mathbf{C}$, $\mathbf{c}_a$ the 3 corresponding orthonormal eigenvectors and $\mathbf{c}_a \otimes \mathbf{c}_a$ the 3 resulting self-dyads, an *isotropic strain* is equivalently characterized by

$$\begin{aligned}
&\mathbf{G}(\gamma_c, \mathbf{c}_c) = g_a(\gamma_1,\gamma_2,\gamma_3)\mathbf{c}_a \otimes \mathbf{c}_a \\
&g_a(1,1,1) = 0
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\nabla_{\mathbf{C}}\mathbf{G}(\gamma_c,\mathbf{c}_c) = \frac{\partial g_a}{\partial \gamma_b}[\mathbf{c}_a\otimes\mathbf{c}_a\otimes\mathbf{c}_b\otimes\mathbf{c}_b] + g_a\nabla_{\mathbf{C}}[\mathbf{c}_a\otimes\mathbf{c}_a] \succ 0 \\
&\frac{\partial g_a}{\partial \gamma_b}(1,1,1) = \frac{1}{2}\delta_{ab}
\end{aligned} \tag{8}$$

where the g_a are 3 cyclically symmetric functions of the γ_b linked to the G_i by $g_a(\gamma_1,\gamma_2,\gamma_3) = G_1[(\gamma_1^2+\gamma_2^2+\gamma_3^2)/2, \gamma_1+\gamma_2+\gamma_3, \ln(\gamma_1\gamma_2\gamma_3)]\gamma_a + G_0[\ldots] + G_{-1}[\ldots]1/\gamma_a$ and their gradients calculated via $\nabla_{\mathbf{C}}g_a = (\partial g_a/\partial\gamma_b)\nabla_{\mathbf{C}}\gamma_b = (\partial g_a/\partial\gamma_b)\mathbf{c}_b\otimes\mathbf{c}_b$ with $\partial g_a/\partial\gamma_b = \partial g_b/\partial\gamma_a$, $a,b = 1,3$. Consequently, the strain principal directions coincide with the metric ones $\mathbf{c}_a$, i.e. $\mathbf{G}(\mathbf{C})$ and $\mathbf{C}$ are *coaxial*. The spectral form (7) resembles the invariant form (4) once the eigenvalues are regarded as invariants and the self-dyads as generators. The trouble with eigenvalues is that they can be repeated, in which case the orthonormal triad of eigenvectors becomes rotationally loose and the actual calculation of the gradient delicate (cf. Hill, 1968; Ball, 1984). This is why the invariant formulas (4) and (5) are computationally preferable to the spectral ones (7) and (8). All classical strains are isotropic strains.

2.4. Simple strain

At this stage, another simplification is introduced, either in the invariant form (4) and (5) or in the spectral one (7) and (8).

The strain gradient (5) is composed of a *dyadic* part $(\partial G_i/\partial \Gamma_j)\mathbf{C}^i \otimes \mathbf{C}^j$ and a *diagonal* part $G_k \nabla_{\mathbf{C}} \mathbf{C}^k$. Since the dyadic part vanishes in the original form and is absent in all classical strains, it is inferred that a subclass of *simple strains* results if it vanishes everywhere. A sufficient condition for that clearly is $\partial G_i/\partial \Gamma_j = 0$, i.e. $G_i(\Gamma_j) = G_i =$ *constant*. In other words, a trinomial Laurent's series with constant coefficients G_i is a simple strain candidate

$$\mathbf{G}(\mathbf{C}) = G_1\mathbf{C} + G_0\mathbf{I} + G_{-1}\mathbf{C}^{-1}, \quad G_1 + G_0 + G_{-1} = 0 \tag{9}$$

$$\nabla_{\mathbf{C}}\mathbf{G}(\mathbf{C}) = G_1\mathbf{I} \,\overline{\underline{\otimes}}\, \mathbf{I} - G_{-1}\mathbf{C}^{-1} \,\overline{\underline{\otimes}}\, \mathbf{C}^{-1} \succ 0, \quad G_1 - G_{-1} = 1/2 \tag{10}$$

It is a nonlinear dilatation of $\mathbf{C}$ that exhibits no *coupling* between principal strains.

Simple strains can also be derived from the spectral form (7) by assuming it to be a simple tensor function, meaning again a nonlinear dilatation of the metric tensor $\mathbf{C}$ (Hill, 1968; Ogden, 1974; Xiao et al., 1998a)

$$\mathbf{G}(\gamma_c, \mathbf{c}_c) = g(\gamma_a)\mathbf{c}_a \otimes \mathbf{c}_a, \quad g(1) = 0 \tag{11}$$

$$\nabla_{\mathbf{C}}\mathbf{G}(\gamma_c, \mathbf{c}_c) = g^{\Delta}(\gamma_a, \gamma_b)\frac{1}{4}[\mathbf{c}_a \otimes \mathbf{c}_b + \mathbf{c}_b \otimes \mathbf{c}_a] \otimes [\mathbf{c}_a \otimes \mathbf{c}_b + \mathbf{c}_b \otimes \mathbf{c}_a] \succ 0 \tag{12}$$

$$g^{\Delta}(\gamma_a, \gamma_b) = \begin{cases} \dfrac{g(\gamma_b) - g(\gamma_a)}{\gamma_b - \gamma_a} & \text{if } \gamma_a \neq \gamma_b \\ g'(\gamma_a) & \text{if } \gamma_a = \gamma_b \end{cases}, \quad g'(\gamma) > 0, \; g'(1) = 1/2$$

Simple scaling guarantees *principal strain uncoupling*. Classical examples of simple functions are polynomials, Laurent's series, the logarithm,... (Moreau, 1979). All classical strains are simple strains.

2.5. Metric strain

Both invariant and spectral reductions (9) and (11) suggest that a Laurent series truncated to its 3 central terms with constant coefficients represents a promising strain prototype. Enforcing the consistency conditions $G_1 + G_0 + G_{-1} = 0$ and $G_1 - G_{-1} = 1/2$ then yields the proposed concept.

A *metric material strain* $\mathbf{E}_n$ ($\equiv\mathbf{G}$) is defined as a smooth monotone isotropic simple function in the form of a linear combination of the three consecutive powers $\mathbf{C}$, $\mathbf{I}$ and $\mathbf{C}^{-1}$

$$\mathbf{E}_n(\mathbf{C}) = \frac{2+n}{8}\mathbf{C} - \frac{n}{4}\mathbf{I} - \frac{2-n}{8}\mathbf{C}^{-1}, \quad \mathbf{E}_n(\mathbf{I}) = \mathbf{O} \quad (-2 \leqslant n \leqslant 2) \tag{13}$$

$$\nabla_{\mathbf{C}}\mathbf{E}_n(\mathbf{C}) = \frac{2+n}{8}\mathbf{I} \,\overline{\underline{\otimes}}\, \mathbf{I} + \frac{2-n}{8}\mathbf{C}^{-1} \,\overline{\underline{\otimes}}\, \mathbf{C}^{-1}, \nabla_{\mathbf{C}}\mathbf{E}_n(\mathbf{I}) = \underline{\underline{\mathbb{I}}}/2 \tag{14}$$

A metric strain can also be viewed as a convex combination of the quadratic and quadhyperbolic strains

$$\mathbf{E}_n = \frac{2+n}{4}\mathbf{E}_2 + \frac{2-n}{4}\mathbf{E}_{-2} \tag{15}$$

The metric strains (13) form a one-parameter family called the *metric* family. It includes an infinity of members since n is real. Focussing attention on integer values of n, the quadratic Green and quadhyperbolic

Index n	Metric strain $\mathbf{E}_n(\mathbf{C})$	Qualifier (**U**-degree)
2	$\mathbf{E}_2(\mathbf{C}) = \frac{1}{2}(\mathbf{C} - \mathbf{I})$	Quadratic
1	$\mathbf{E}_1(\mathbf{C}) = \frac{1}{8}(3\mathbf{C} - 2\mathbf{I} - \mathbf{C}^{-1})$	Quasilinear
0	$\mathbf{E}_0(\mathbf{C}) = \frac{1}{4}(\mathbf{C} - \mathbf{C}^{-1})$	Quasilogarithmic
−1	$\mathbf{E}_{-1}(\mathbf{C}) = \frac{1}{8}(\mathbf{C} + 2\mathbf{I} - 3\mathbf{C}^{-1})$	Quasihyperbolic
−2	$\mathbf{E}_{-2}(\mathbf{C}) = \frac{1}{2}(\mathbf{I} - \mathbf{C}^{-1})$	Quadhyperbolic

Karni strains are recovered for the two extreme values $n = \pm 2$, the quasilogarithmic strain for the middle value $n = 0$ (Pietrzak, 1997; Pietrzak and Curnier, 1999) and two new strains are uncovered for $n = \pm 1$.

In fact, the metric family is a second-order approximation in terms of $\|\mathbf{U}\|$ of the *Seth–Hill stretch family* (Doyle and Ericksen, 1956; Seth, 1964; Hill, 1968)

$$\underline{\mathbf{E}}_m(\mathbf{U}) = \frac{1}{m}(\mathbf{U}^m - \mathbf{I}), \quad \underline{\mathbf{E}}_0(\mathbf{U}) = \ln \mathbf{U} \qquad (-2 \leqslant m \leqslant 2) \tag{16}$$

While the extremes are the same $\mathbf{E}_{\mp 2}(\mathbf{U}^2) = \underline{\mathbf{E}}_{\mp 2}(\mathbf{U})$, the intermediates are close

$$\mathbf{E}_1(\mathbf{U}^2) = \frac{1}{8}(3\mathbf{U}^2 - 2\mathbf{I} - \mathbf{U}^{-2}) \approx \underline{\mathbf{E}}_1(\mathbf{U}) = \mathbf{U} - \mathbf{I}$$

$$\mathbf{E}_0(\mathbf{U}^2) = \frac{1}{4}(\mathbf{U}^2 - \mathbf{U}^{-2}) \approx \underline{\mathbf{E}}_0(\mathbf{U}) = \ln \mathbf{U}$$

$$\mathbf{E}_{-1}(\mathbf{U}^2) = \frac{1}{8}(\mathbf{U}^2 + 2\mathbf{I} - 3\mathbf{U}^{-2}) \approx \underline{\mathbf{E}}_{-1}(\mathbf{U}) = \mathbf{I} - \mathbf{U}^{-1}$$

hence their names.

Injecting the spectral decomposition (6) of $\mathbf{C}$ into (13) gives its spectral form

$$\mathbf{E}_n(\gamma_c, \mathbf{c}_c) = e_n(\gamma_a)\mathbf{c}_a \otimes \mathbf{c}_a \tag{17}$$

$$e_n(\gamma) = \frac{2+n}{8}\gamma - \frac{n}{4} - \frac{2-n}{8}\frac{1}{\gamma}, \quad e_n(1) = 0$$

$$\nabla_{\mathbf{C}}\mathbf{E}_n(\gamma_c, \mathbf{c}_c) = e_n'(\sqrt{\gamma_a \gamma_b})\frac{1}{4}[\mathbf{c}_a \otimes \mathbf{c}_b + \mathbf{c}_b \otimes \mathbf{c}_a] \otimes [\mathbf{c}_a \otimes \mathbf{c}_b + \mathbf{c}_b \otimes \mathbf{c}_a] \succ 0 \tag{18}$$

$$e_n'(\gamma) = \frac{2+n}{8} + \frac{2-n}{8}\frac{1}{\gamma^2} > 0, \quad e_n'(1) = 1/2$$

The metric scale e_n is a monotone function of γ and therefore the metric strain $\mathbf{E}_n$ is a monotone function of $\mathbf{C}$. The graphs of the integer scale functions e_n are plotted in terms of the stretch $\upsilon = \sqrt{\gamma}$ in Fig. 1, for comparison. They are representative of a simple elongation of a bar. The *quasilinear* strain $\mathbf{E}_1$ is outstanding because its curvature vanishes and its inflexion occurs at the origin. Over their extended domain $0 \leqslant \upsilon \leqslant \infty$,

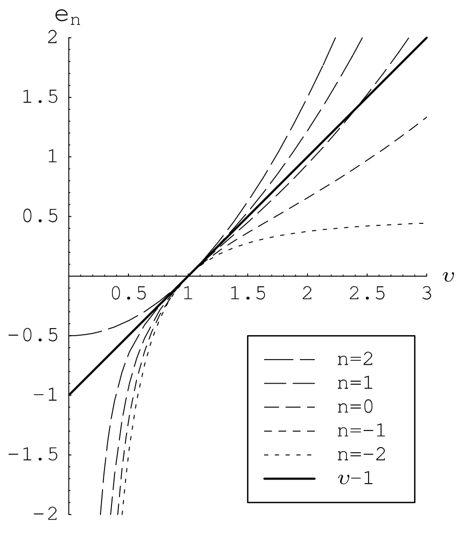

Fig. 1. Graphs $e = e_n(\upsilon)$ of metric strains versus stretch.

the scales $e_{\pm 2}$: $\upsilon \mapsto e_{\pm 2}(\upsilon)$ of the quadratic and quadhyperbolic strains reach the limits: $e_2(0) = -1/2$ ($e'_2(0) = 0$) and $e_{-2}(\infty) = 1/2$ ($e'_{-2}(\infty) = 0$). These limits indicate that e_2 and e_{-2} are strictly monotone over $0 < \gamma < \infty$ (as they must) but no longer at 0 and ∞, respectively. Consequently, $\mathbf{E}_{\pm 2}$ are *not coercive*. This confirms that $\mathbf{E}_2$ and $\mathbf{E}_{-2}$ are upper and lower bounds, respectively, for other strains and that n must be kept within its preassigned range $-2 \leqslant n \leqslant 2$.

2.6. Nominal-metric strain

Using the metric tensor definition $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, the generalized, isotropic, simple and metric strains can be expressed in terms of the nominal strain $\mathbf{F}$. In particular, the metric strain $\mathbf{E}_n$ and its non-symmetric gradient $\nabla_{\mathbf{F}}\mathbf{E}_n (\neq \nabla^{\mathrm{T}}_{\mathbf{F}}\mathbf{E}_n)$ are equal to

$$\mathbf{E}_n(\mathbf{F}) = \frac{2+n}{8}\mathbf{F}^{\mathrm{T}}\mathbf{F} - \frac{n}{4}\mathbf{I} - \frac{2-n}{8}\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}} \tag{19}$$

$$\nabla_{\mathbf{F}}\mathbf{E}_n(\mathbf{F}) = \frac{2+n}{8}[\mathbf{F}^{\mathrm{T}} \underline{\otimes} \mathbf{I} + \mathbf{I} \overline{\otimes} \mathbf{F}^{\mathrm{T}}] + \frac{2-n}{8}[\mathbf{F}^{-1} \underline{\otimes} (\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}) + (\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}) \overline{\otimes} \mathbf{F}^{-1}] \tag{20}$$

2.7. Metric strain rate

For a generalized strain (1), the rate is the composition of the metric rate by the strain gradient

$$\dot{\mathbf{G}}(\dot{\mathbf{C}}, \mathbf{C}) = \nabla_{\mathbf{C}}\mathbf{G}(\mathbf{C})\dot{\mathbf{C}} \tag{21}$$

Note that $\dot{\mathbf{G}}(\mathbf{O}, \mathbf{C}) = \mathbf{O}$ and $\dot{\mathbf{G}}(\dot{\mathbf{C}}, \mathbf{I}) = \dot{\mathbf{C}}/2$ as expected. By construction, this linear relationship is invertible as: $\dot{\mathbf{C}}(\dot{\mathbf{G}}, \mathbf{C}) = \nabla^{-1}_{\mathbf{C}}\mathbf{G}(\mathbf{C})\dot{\mathbf{G}}$.

The *rate* of a metric strain is easily found in terms of the metric rate $\dot{\mathbf{C}}$ as

$$\dot{\mathbf{E}}_n(\dot{\mathbf{C}}, \mathbf{C}) = \frac{2+n}{8}\dot{\mathbf{C}} + \frac{2-n}{8}\mathbf{C}^{-1}\dot{\mathbf{C}}\mathbf{C}^{-1} = \left[\frac{2+n}{8}\mathbf{I} \underline{\overline{\otimes}} \mathbf{I} + \frac{2-n}{8}\mathbf{C}^{-1} \underline{\overline{\otimes}} \mathbf{C}^{-1}\right]\dot{\mathbf{C}} = [\nabla_{\mathbf{C}}\mathbf{E}_n(\mathbf{C})]\dot{\mathbf{C}} \tag{22}$$

Of course, $\dot{\mathbf{E}}_n(\mathbf{O}, \mathbf{C}) = \mathbf{O}$ and $\dot{\mathbf{E}}_n(\dot{\mathbf{C}}, \mathbf{I}) = \dot{\mathbf{C}}/2$. Therefore, the metric strain rates are much simpler to calculate than the rates of the stretch family (16) which remain complicate in spite of the many attempts to simplify them (Hill, 1978; Fitzgerald, 1980; Ball, 1984; Hoger and Carlson, 1984a,b; Carlson and Hoger, 1986; Guo, 1984; Curnier and Rakotomanana, 1991; Scheidler, 1991; Man and Guo, 1993; Xiao et al., 1998b; Guan-Suo et al., 1999; Rosati, 2000).

2.8. Nominal-metric strain rate

Using the basic formula

$$\dot{\mathbf{C}} = \mathbf{F}^{\mathrm{T}}\dot{\mathbf{F}} + \dot{\mathbf{F}}^{\mathrm{T}}\mathbf{F} = [\mathbf{F}^{\mathrm{T}} \underline{\otimes} \mathbf{I} + \mathbf{I} \overline{\otimes} \mathbf{F}^{\mathrm{T}}]\dot{\mathbf{F}} = [\nabla_{\mathbf{F}}\mathbf{C}]\dot{\mathbf{F}} \tag{23}$$

all the generalized, isotropic, simple and metric strain rates can in turn be expressed in terms of the nominal strain rate $\dot{\mathbf{F}}$. In particular, the *metric strain rate* is equal to

$$\begin{aligned}\dot{\mathbf{E}}_n(\dot{\mathbf{F}}, \mathbf{F}) &= [\nabla_{\mathbf{F}}\mathbf{E}_n(\mathbf{F})]\dot{\mathbf{F}} = \frac{2+n}{8}(\mathbf{F}^{\mathrm{T}}\dot{\mathbf{F}} + \dot{\mathbf{F}}^{\mathrm{T}}\mathbf{F}) + \frac{2-n}{8}(\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}} + \mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}\dot{\mathbf{F}}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}) \\ &= \left\{\frac{2+n}{8}[\mathbf{F}^{\mathrm{T}} \underline{\otimes} \mathbf{I} + \mathbf{I} \overline{\otimes} \mathbf{F}^{\mathrm{T}}] + \frac{2-n}{8}[\mathbf{F}^{-1} \underline{\otimes} (\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}) + (\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}) \overline{\otimes} \mathbf{F}^{-1}]\right\}\dot{\mathbf{F}}\end{aligned} \tag{24}$$

3. Metric stress

The concept of generalized stress was introduced by Ziegler and MacVean (1967), MacVean (1968), confirmed by Hill (1968, 1978) and studied by many others (Guo and Dubey, 1984; Atluri, 1984; Curnier and Rakotomanana, 1984; Billington, 1985, 1986; Curnier and Rakotomanana, 1991; Xiao, 1995).

3.1. Metric stress provisions

For the same computational facility reason as for a strain and in view of the classical stresses, a generalized stress $\mathbf{Z}$ is looked for in terms of the Kirchhoff material stress $\mathbf{S}$ and the metric $\mathbf{C}$ (rather than the stretch $\mathbf{U}$), i.e. in a form $\mathbf{Z}(\mathbf{S},\mathbf{C})$. Moreover, in view of the same classical stresses, a generalized stress is further restricted to a *linear* function of the material stress, hence $\mathbf{Z}(\mathbf{S},\mathbf{C}) = \mathbb{Z}(\mathbf{C})\mathbf{S}$.

3.2. Generalized stress

A *generalized material stress* $\mathbf{Z}$ is defined as a symmetric tensor valued monotone linear function of the symmetric material stress tensor $\mathbf{S}$ with a $\mathbf{C}$-dependent gradient, which coincides with $\mathbf{S}$ in the original form Ω i.e. with a unit gradient there

$$\mathbf{Z} : (\mathbf{S},\mathbf{C}) \mapsto \mathbf{Z}(\mathbf{S},\mathbf{C}) = \mathbb{Z}(\mathbf{C})\mathbf{S}, \quad \mathbf{Z}(\mathbf{S},\mathbf{I}) = \mathbf{S} \tag{25}$$

$$\nabla_{\mathbf{S}}\mathbf{Z}(\mathbf{S},\mathbf{C}) = \mathbb{Z}(\mathbf{C}) = \mathbb{Z}^{\mathrm{T}}(\mathbf{C}) \succ 0, \quad \mathbb{Z}(\mathbf{I}) = \underline{\mathbb{I}} \tag{26}$$

Dependence on $\mathbf{S}$ and $\mathbf{C}$ guarantees *objectivity*. Monotone linearity in $\mathbf{S}$ guarantees *bijectivity* between $\mathbf{Z}(\mathbf{S},\mathbf{C})$ and $\mathbf{S}$ for all $\mathbf{C}$ and hereby the existence of an *inverse* $\mathbf{S}(\mathbf{Z},\mathbf{C}) = \mathbb{Z}^{-1}(\mathbf{C})\mathbf{Z}$. Major symmetry of $\mathbb{Z}$ guarantees *stress path indifference*. Finally, reduction of $\mathbb{Z}$ to the identity in the original form guarantees that a generalized stress $\mathbf{Z}$ converges to $\mathbf{S}$ (and $\mathbf{P}$) for sufficiently small strains.

For a straightforward calculation of the nominal stress $\mathbf{P} = \mathbf{FS}$ and subsequent formulation of the boundary value problem, a generalized stress is better formulated in its *partial inverse form*

$$\mathbf{S} : (\mathbf{Z},\mathbf{C}) \mapsto \mathbf{S}(\mathbf{Z},\mathbf{C}) = \mathbb{S}(\mathbf{C})\mathbf{Z}, \quad \mathbf{S}(\mathbf{Z},\mathbf{I}) = \mathbf{Z} \tag{27}$$

$$\nabla_{\mathbf{Z}}\mathbf{S}(\mathbf{Z},\mathbf{C}) = \mathbb{S}(\mathbf{C}) = \mathbb{S}^{\mathrm{T}}(\mathbf{C}) \succ 0, \quad \mathbb{S}(\mathbf{I}) = \underline{\mathbb{I}} \tag{28}$$

3.3. Isotropic stress

In addition to (27) and (28), a generalized stress $\mathbf{Z}$ is required to be an *isotropic* function of both $\mathbf{S}$ and $\mathbf{C}$ for avoiding the introduction of an artificial static and geometric anisotropy

$$\mathbf{S}(\mathbf{RZR}^{\mathrm{T}},\mathbf{RCR}^{\mathrm{T}}) = \mathbf{RS}(\mathbf{Z},\mathbf{C})\mathbf{R}^{\mathrm{T}} \quad \forall \mathbf{R} = \mathbf{R}^{-\mathrm{T}} \tag{29}$$

Isotropy guarantees *material direction indifference* of the stress measure. The theorem of representation of an isotropic function of two symmetric tensors $\mathbf{Z}$ and $\mathbf{C}$ then provides the general form of a ($\mathbf{Z}$-nonlinear) isotropic stress as a combination of 8 generators with coefficients depending on 10 mixed invariants of $\mathbf{Z}$ and $\mathbf{C}$, including the 3 pure invariants Γ_k of $\mathbf{C}$ (those 8 generators are the 8 partial $\mathbf{Z}$- and $\mathbf{C}$-gradients of these 10 invariants). Letting the 3 invariants of $\mathbf{C}$ aside and deleting the cubic invariant $\mathrm{tr}^3\mathbf{Z}/3$ of $\mathbf{Z}$ in order to retrieve a $\mathbf{Z}$-linear stress representation, the remaining 6 invariants Σ_i (3 linear and 3 quadratic), together with their $\mathbf{Z}$-gradients, are

Σ_i	$\mathrm{tr}(\mathbf{CZ})$	$\mathrm{tr}\mathbf{Z}$	$\mathrm{tr}(\mathbf{C}^{-1}\mathbf{Z})$	$\frac{1}{2}\mathrm{tr}(\mathbf{CZ})^2$	$\frac{1}{2}\mathrm{tr}\mathbf{Z}^2$	$\frac{1}{2}\mathrm{tr}(\mathbf{C}^{-1}\mathbf{Z})^2$
$\nabla_{\mathbf{Z}}\Sigma_i$	$\mathbf{C}$	$\mathbf{I}$	$\mathbf{C}^{-1}$	$\mathbf{CZC}$	$\mathbf{Z}$	$\mathbf{C}^{-1}\mathbf{Z}\mathbf{C}^{-1}$
$\nabla^2_{\mathbf{Z}}\Sigma_i$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbf{C}\,\overline{\underline{\otimes}}\,\mathbf{C}$	$\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}$	$\mathbf{C}^{-1}\,\overline{\underline{\otimes}}\,\mathbf{C}^{-1}$

More specifically, an *isotropic material stress* $\mathbf{Z}$ is defined as the partial inverse of a smooth monotone isotropic function of the reference stress $\mathbf{S}$ and the metric $\mathbf{C}$, which is linear in the stress $\mathbf{S}$ and based on the three metric powers $\mathbf{C}$, $\mathbf{I}$ and $\mathbf{C}^{-1}$ and which coincides with the reference stress in the original form Ω,

$$\mathbf{S}(\mathbf{Z},\mathbf{C}) = S_{ij}(\Gamma_1,\Gamma_0,\Gamma_{-1})\mathrm{tr}(\mathbf{C}^j\mathbf{Z})\mathbf{C}^i + S_k(\Gamma_1,\Gamma_0,\Gamma_{-1})\mathbf{C}^k\mathbf{Z}\mathbf{C}^k \tag{30}$$

$$\nabla_{\mathbf{Z}}\mathbf{S}(\mathbf{Z},\mathbf{C}) = \mathbb{S}(\mathbf{C}) = S_{ij}\mathbf{C}^i\otimes\mathbf{C}^j + S_k\mathbf{C}^k\,\overline{\underline{\otimes}}\,\mathbf{C}^k \tag{31}$$

$$\Sigma_i\Sigma_j S_{ij}(3/2,3,0) = 0$$

$$\Sigma_k S_k(3/2,3,0) = S_1(3/2,3,0) + S_0(3/2,3,0) + S_{-1}(3/2,3,0) = 1$$

In these equations, S_{ij} and S_k $(i,j,k = 1,0,-1)$ are 12 functions of the metric invariants Γ_i (due to the $\mathbf{Z}$-linearity assumption). This figure drops down to 9 in view of the symmetry of the stress gradient (31) and thus of the coefficient matrix: $S_{ji} = S_{ij}$ (which is necessary and sufficient for stress path indifference). Sufficient conditions for $\Sigma_i\Sigma_j\,{}_{\mathrm{S}ij}(3/2,3,0) = 0$ are $S_{ij}(3/2,3,0) = 0$ and a fortiori $S_{ij}(\Gamma_1,\Gamma_0,\Gamma_{-1}) = 0$.

Using the spectral decomposition (6) of the metric tensor and the property $[\mathbf{c}_a\otimes\mathbf{c}_a]\,\overline{\underline{\otimes}}\,[\mathbf{c}_b\otimes\mathbf{c}_b] = \mathbf{c}_a\otimes\mathbf{c}_b \otimes\mathbf{c}_a\otimes\mathbf{c}_b$, the stress gradient (31) can be equivalently written

$$\nabla_{\mathbf{Z}}\mathbf{S}(\mathbf{Z},\mathbf{C}) = S_{ij}\gamma_a^i\gamma_b^j\mathbf{c}_a\otimes\mathbf{c}_a\otimes\mathbf{c}_b\otimes\mathbf{c}_b + S_k\gamma_a^k\gamma_b^k\mathbf{c}_a\otimes\mathbf{c}_b\otimes\mathbf{c}_a\otimes\mathbf{c}_b \tag{32}$$

where $a,b = 1,3$ and $i,j = 1,0,-1$.

3.4. Simple stress

Observing that the stress gradient (31) is made of a *dyadic* part $S_{ij}\mathrm{tr}(\mathbf{C}^j\mathbf{Z})\mathbf{C}^i$ which vanishes at the origin and a *diagonal* part $S_k\mathbf{C}^k\mathbf{Z}\mathbf{C}^k$, it is inferred that a subclass of simple stresses will result if its dyadic part vanishes everywhere, as for simple strains. A sufficient condition for a zero dyadic gradient clearly is $S_{ij} = 0$.

Consequently, a *simple stress* is introduced as a partial inverse (linear, monotone) diagonal isotropic stress

$$\mathbf{S}(\mathbf{Z},\mathbf{C}) = S_1\mathbf{CZC} + S_0\mathbf{Z} + S_{-1}\mathbf{C}^{-1}\mathbf{Z}\mathbf{C}^{-1} \tag{33}$$

$$\nabla_{\mathbf{Z}}\mathbf{S}(\mathbf{Z},\mathbf{C}) = S_1\mathbf{C}\,\overline{\underline{\otimes}}\,\mathbf{C} + S_0\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I} + S_{-1}\mathbf{C}^{-1}\,\overline{\underline{\otimes}}\,\mathbf{C}^{-1} \tag{34}$$

$$S_1(3/2,3,0) + S_0(3/2,3,0) + S_{-1}(3/2,3,0) = 1$$

It is a nonlinear dilatation of the stress tensor that shows no *coupling* between principal stresses.

By a similar hypothesis, the stress gradient in spectral form (32) can be reduced to the more simple form

$$\nabla_{\mathbf{Z}}\mathbf{S}(\mathbf{Z},\mathbf{C}) = S_k\gamma_a^k\gamma_b^k\mathbf{c}_a\otimes\mathbf{c}_b\otimes\mathbf{c}_a\otimes\mathbf{c}_b \tag{35}$$

Simple scaling guarantees *principal stress uncoupling*. All classical stresses are simple stresses.

3.5. Metric stress

A look at the classical stresses, while keeping in mind the bounding roles of the Kirchhoff and Rivlin stresses, further suggests to select the constant coefficients $S_1 = 0$, $S_0 + S_{-1} = 1$, $(0 \leqslant S_0, S_{-1} \leqslant 1)$, hence to define $\mathbf{S}$ as a convex combination of $\mathbf{Z}$ and $\mathbf{C}^{-1}\mathbf{Z}\mathbf{C}^{-1}$.

A *metric material stress* $\mathbf{S}_n(\equiv\mathbf{Z})$ is defined as the partial inverse of a smooth monotone isotropic simple function of the reference stress $\mathbf{S}$ and the metric $\mathbf{C}$, which is linear in $\mathbf{S}$ and a linear combination of the three successive powers $\mathbf{C}$, $\mathbf{I}$ and $\mathbf{C}^{-1}$ and which coincides with the reference stress in the original form,

$$\mathbf{S}(\mathbf{S}_n,\mathbf{C}) = \frac{2+n}{4}\mathbf{S}_n + \frac{2-n}{4}\mathbf{C}^{-1}\mathbf{S}_n\mathbf{C}^{-1}, \quad \mathbf{S}(\mathbf{S}_n,\mathbf{I}) = \mathbf{S}_n \quad (-2 \leqslant n \leqslant 2) \tag{36}$$

$$\nabla_{\mathbf{S}_n}\mathbf{S}(\mathbf{S}_n,\mathbf{C}) = \frac{2+n}{4}\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} + \frac{2-n}{4}\mathbf{C}^{-1}\,\underline{\overline{\otimes}}\,\mathbf{C}^{-1}, \quad \nabla_{\mathbf{S}_n}\mathbf{S}(\mathbf{S}_n,\mathbf{I}) = \underline{\mathbb{I}} \tag{37}$$

Injecting the spectral decomposition (6) of $\mathbf{C}$ into (36) gives its spectral form

$$\mathbf{S}(\mathbf{S}_n,\mathbf{C}) = \frac{2+n}{4}\mathbf{S}_n + \frac{2-n}{4\gamma_a\gamma_b}[\mathbf{c}_a\otimes\mathbf{c}_a]\mathbf{S}_n[\mathbf{c}_b\otimes\mathbf{c}_b] \tag{38}$$

The metric stresses (36) form a one-parameter family called the *metric* family. It includes an infinity of members because n is real.

Index n	Metric stress $\mathbf{S}(\mathbf{S}_n,\mathbf{C})$	Qualifier ($\mathbf{U}$-degree)
2	$\mathbf{S}(\mathbf{S}_2,\mathbf{C}) = \mathbf{S}(\mathbf{S}_2) \equiv \mathbf{S}_2$	Quadratic
1	$\mathbf{S}(\mathbf{S}_1,\mathbf{C}) = \frac{1}{4}(3\mathbf{S}_1 + \mathbf{C}^{-1}\mathbf{S}_1\mathbf{C}^{-1})$	Quasilinear
0	$\mathbf{S}(\mathbf{S}_0,\mathbf{C}) = \frac{1}{2}(\mathbf{S}_0 + \mathbf{C}^{-1}\mathbf{S}_0\mathbf{C}^{-1})$	Quasilogarithmic
−1	$\mathbf{S}(\mathbf{S}_{-1},\mathbf{C}) = \frac{1}{4}(\mathbf{S}_{-1} + 3\mathbf{C}^{-1}\mathbf{S}_{-1}\mathbf{C}^{-1})$	Quasihyperbolic
−2	$\mathbf{S}(\mathbf{S}_{-2},\mathbf{C}) = \mathbf{C}^{-1}\mathbf{S}_{-2}\mathbf{C}^{-1}$	Quadhyperbolic

The classical Kirchhoff and Rivlin stresses are recovered for the two extreme values $n = \pm 2$, the quasi-logarithmic stress for the middle value $n = 0$ (Pietrzak, 1997; Pietrzak and Curnier, 1999) and two new stresses are uncovered for $n = \pm 1$.

The metric stress family is an approximation of the stretch stress family. In particular, the new intermediate stresses $\mathbf{S}_{\pm 1}$ are simple approximations of the corresponding *Biot* (–Ziegler) linear and *Hill* hyperbolic ones $\underline{\mathbf{S}}_{\pm 1}$, hence their names

$$\mathbf{S}_1(\mathbf{S},\mathbf{U}^2) \approx \underline{\mathbf{S}}_1(\mathbf{S},\mathbf{U}), \quad \mathbf{S}_{-1}(\mathbf{S},\mathbf{U}^2) \approx \underline{\mathbf{S}}_{-1}(\mathbf{S},\mathbf{U})$$

3.6. Conjugacy definition

A weakness of the static definition of metric stresses is its failure to reveal a deep correspondence with its homonymous strain. This correspondence, called *duality* or *conjugacy*, arises from the additional, rational, requirement that all strain–stress pairs must develop the *same internal power* (on any part $\omega \subseteq \Omega$ of the solid), in order to be energetically equivalent. Since the internal power implied in deforming a solid must in turn be equal to the external power supplied to it, the above requirement, written in the preferred nominal-material description, takes the form

$$\int_\omega \mathbf{Z} : \dot{\mathbf{G}}\,\mathrm{d}V = \int_\omega \mathbf{P} : \dot{\mathbf{F}}\,\mathrm{d}V = \int_{\partial\omega} \mathbf{p}\cdot\dot{\mathbf{y}}\,\mathrm{d}A \quad \forall\omega \subseteq \Omega \tag{39}$$

where $\mathbf{P} : \dot{\mathbf{F}} = \mathrm{tr}(\mathbf{P}^{\mathrm{T}}\dot{\mathbf{F}})$ denotes the stress–strain duality product which reduces to $\mathbf{Z} : \dot{\mathbf{G}} = \mathrm{tr}(\mathbf{Z}\dot{\mathbf{G}})$ for material symmetric tensors. Assuming continuity of the internal power densities with respect to the original position $\mathbf{x}$, a generalized strain–stress pair $\mathbf{G}$–$\mathbf{Z}$ is said to be *conjugate* if and only if

$$\mathbf{Z} : \dot{\mathbf{G}} = \mathbf{P} : \dot{\mathbf{F}} \quad (\mathbf{G}^{\mathrm{T}} = \mathbf{G}, \mathbf{Z}^{\mathrm{T}} = \mathbf{Z}) \tag{40}$$

In particular, it is known (cf. Truesdell and Toupin, 1960; Truesdell and Noll, 1965; Eringen, 1975), that the quadratic Green strain and Kirchhoff stress pair **E**–**S** is conjugate. Therefore the material pair **C**/2–**S** can be beneficially used for reference in the definition (40) instead of the nominal pair **F**–**P**. Finally, if the a posteriori principle (40) of invariance of the internal power in a change of strain–stress pair is turned into an a priori postulate, then it can be used for defining stresses once strains are given.

3.7. Conjugate generalized stress

A material, symmetric *generalized stress* **Z** *conjugate* to a given material, symmetric *generalized strain* **G** is implicitly defined by requiring that the internal power (per unit material volume) it develops at the strain rate $\dot{\mathbf{G}}$ must be equal to the reference power developed by the material stress **S** at the material strain rate $\dot{\mathbf{E}} = \dot{\mathbf{C}}/2$

$$\mathbf{Z} : \dot{\mathbf{G}} = \mathbf{S} : \dot{\mathbf{C}}/2 \quad (\mathbf{Z}^{\mathrm{T}} = \mathbf{Z}) \tag{41}$$

Unlike the static notion of generalized stress (27) which is free of any strain connotation, the energetic concept of conjugate generalized stress (41) is linked to a definite strain. This link can be found by substituting the generalized strain rate (21) in the internal power (41) which, with the help of the transposition of a fourth order tensor $\mathbb{A}^{\mathrm{T}} | \ \mathbf{X} : \mathbb{A}\mathbf{Y} = \mathbb{A}^{\mathrm{T}}\mathbf{X} : \mathbf{Y}$, yields

$$\mathbf{Z} : \dot{\mathbf{G}} = \mathbf{Z} : \nabla_{\mathbf{C}}\mathbf{G}(\mathbf{C})\dot{\mathbf{C}} = \nabla_{\mathbf{C}}^{\mathrm{T}}\mathbf{G}(\mathbf{C})\mathbf{Z} : \dot{\mathbf{C}} = (\mathbf{S}/2) : \ \dot{\mathbf{C}}$$

Identifying the duals of $\dot{\mathbf{C}}$ (since it is arbitrary), a *generalized stress* **Z** *conjugate* to a *generalized strain* **G** is alternately defined in terms of the material stress **S** and the metric **C** as the partial inverse of the **Z**-linear formula

$$\mathbf{S}(\mathbf{Z}, \mathbf{C}) = 2\nabla_{\mathbf{C}}^{\mathrm{T}}\mathbf{G}(\mathbf{C})\mathbf{Z} \tag{42}$$

$$\nabla_{\mathbf{Z}}\mathbf{S}(\mathbf{Z}, \mathbf{C}) = \mathbb{S}(\mathbf{C}) = 2\nabla_{\mathbf{C}}^{\mathrm{T}}\mathbf{G}(\mathbf{C}) \tag{43}$$

Since the strain gradient is positive definite, it is invertible and a generalized stress is directly defined by $\mathbf{Z}(\mathbf{S}, \mathbf{C}) = \frac{1}{2}\nabla_{\mathbf{C}}^{-\mathrm{T}}\mathbf{G}(\mathbf{C})\mathbf{S}$.

3.8. Conjugate isotropic stress

Substituting the isotropic strain gradient (5) into the conjugate stress definition (42) gives the general expression of an *isotropic stress conjugate to an isotropic strain*

$$\mathbf{S}(\mathbf{Z}, \mathbf{C}) = 2\frac{\partial G_i}{\partial \Gamma_j}(\Gamma_1, \Gamma_0, \Gamma_{-1})\mathrm{tr}(\mathbf{C}^j\mathbf{Z})\mathbf{C}^i + 2G_k(\Gamma_1, \Gamma_0, \Gamma_{-1})[\nabla_{\mathbf{C}}\mathbf{C}^k]\mathbf{Z} \tag{44}$$

Here also, the conjugate isotropic stress (44) differs from the plain one (30) by the stress combination coefficient functions being equal to twice the strain gradient (5) ones

$$S_{ij} = 2\frac{\partial G_i}{\partial \Gamma_j}, \quad S_1 = 0, \quad S_0 = 2G_1, \quad S_{-1} = -2G_{-1} \tag{45}$$

3.9. Conjugate simple stress

For a simple strain, the strain gradient further simplifies into (10), so that a *simple stress conjugate to a simple strain* is defined by

$$\mathbf{S}(\mathbf{Z}, \mathbf{C}) = 2G_1(\Gamma_1, \Gamma_0, \Gamma_{-1})\mathbf{Z} - 2G_{-1}(\Gamma_1, \Gamma_0, \Gamma_{-1})\mathbf{C}^{-1}\mathbf{Z}\mathbf{C}^{-1} \tag{46}$$

Conjugacy requires the coefficient functions in (33) to be

$$S_1 = 0, \quad S_0 = 2G_1, \quad S_{-1} = -2G_{-1}$$

3.10. Conjugate metric stress

Finally, the same approach applied to a *metric strain* directly gives the *conjugate metric stress* as

$$\mathbf{S}(\mathbf{S}_n, \mathbf{C}) = 2[\nabla^{\mathrm{T}}_{\mathbf{C}}\mathbf{E}_n(\mathbf{C})]\mathbf{S}_n = \left[\frac{2+n}{4}\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I} + \frac{2-n}{4}\mathbf{C}^{-1}\,\overline{\underline{\otimes}}\,\mathbf{C}^{-1}\right]\mathbf{S}_n = \frac{2+n}{4}\mathbf{S}_n + \frac{2-n}{4}\mathbf{C}^{-1}\mathbf{S}_n\mathbf{C}^{-1} \tag{47}$$

Hence the choice of the (constant coefficient) convex combination

$$S_1 = 0, \quad S_0 = 2G_1 = \frac{2+n}{4}, \quad S_{-1} = -2G_{-1} = \frac{2-n}{4}$$

in (36) is confirmed (G_1 and S_0 have shifted indexes because $G_0 = -n/4$ in (13) disappears in (14)).

3.11. Nominal stress

The material stress derived above is a step towards the nominal stress rather than an end. By substituting the material stress $\mathbf{S}(\mathbf{S}_n, \mathbf{C})$ (36) or (47) into the relationship $\mathbf{P} = \mathbf{FS}$, the *nominal stress* $\mathbf{P}$ is found in terms of the metric stress $\mathbf{S}_n$ and the nominal strain $\mathbf{F}$ to be

$$\mathbf{P}(\mathbf{S}_n, \mathbf{F}) = \mathbf{FS}(\mathbf{S}_n, \mathbf{F}^{\mathrm{T}}\mathbf{F}) = \frac{2+n}{4}\mathbf{FS}_n + \frac{2-n}{4}\mathbf{F}^{-\mathrm{T}}\mathbf{S}_n\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}} \tag{48}$$

The nominal stress remains a linear function of the metric stress. It can therefore be represented by a (fourth order) tensor which can be shown by conjugacy to be the transpose of the non-symmetric gradient of $\mathbf{E}_n$ with respect to $\mathbf{F}$

$$\begin{aligned}\mathbf{P}(\mathbf{S}_n, \mathbf{F}) &= \mathbb{P}(\mathbf{F})\mathbf{S}_n = \left[\nabla^{\mathrm{T}}_{\mathbf{F}}\mathbf{E}_n(\mathbf{F})\right]\mathbf{S}_n \\ &= \left\{\frac{2+n}{8}[\mathbf{F}\,\underline{\otimes}\,\mathbf{I} + \mathbf{F}\,\overline{\otimes}\,\mathbf{I}] + \frac{2-n}{8}[\mathbf{F}^{-\mathrm{T}}\,\underline{\otimes}\,(\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}) + \mathbf{F}^{-\mathrm{T}}\,\overline{\otimes}\,(\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}})]\right\}\mathbf{S}_n\end{aligned} \tag{49}$$

To close this section, it is pointed out that the use of a conjugate strain–stress pair is by no means compulsory for formulating material laws. It is only preferable for energetic balance purposes.

4. Metric elasticity

The metric strain–stress family opens the way for formulating gradual families of nominal laws at large transformations. This capability will now be demonstrated in elasticity.

4.1. Linear metric law

To this end, consider a *hyperelastic linear metric law* $\mathbf{S}_n$, which derives from a *quadratic elastic metric energy density* V_n and possesses a *constant gradient* called the *metric stiffness elasticity tensor* $\mathbb{S}_n$,

$$V_n(\mathbf{E}_n) = \frac{1}{2}\mathbf{E}_n : \mathbb{S}_n\mathbf{E}_n, \qquad V_n(\mathbf{O}) = 0 \tag{50}$$

$$\mathbf{S}_n(\mathbf{E}_n) = \nabla_{\mathbf{E}_n}V_n(\mathbf{E}_n) = \mathbb{S}_n\mathbf{E}_n, \qquad \mathbf{S}_n(\mathbf{O}) = \mathbf{O} \tag{51}$$

$$\mathbb{S}_n = \nabla_{\mathbf{E}_n}\mathbf{S}_n = \nabla^2_{\mathbf{E}_n}V_n, \qquad \mathbb{S}_n \succ 0 \tag{52}$$

The energy V_n and the stress $\mathbf{S}_n$ are assumed to vanish in the original-natural form Ω, without loss of generality and for simplicity, respectively. The stiffness tensor $\mathbb{S}_n$ possesses the minor symmetries resulting from those of $\mathbf{E}_n$ and $\mathbf{S}_n$ and the major symmetry due to the existence of V_n.

For stability reasons in small and pure symmetric strain situations, it is assumed that the *metric* energy function V_n is strictly *convex*, or, equivalently, the stress law $\mathbf{S}_n$ strictly *monotone*, or, sufficiently, the stiffness tensor $\mathbb{S}_n$ *positive* definite over $\mathscr{S}ym = \{\mathbf{E}_n, \mathbf{E}_n^{\mathrm{T}} = \mathbf{E}_n\}$, i.e. $\forall \mathbf{E}_n, \widetilde{\mathbf{E}}_n \in \mathscr{S}ym, \mathbf{E}_n \neq \widetilde{\mathbf{E}}_n$

$$V_n[\alpha\widetilde{\mathbf{E}}_n + (1-\alpha)\mathbf{E}_n] < \alpha V_n(\widetilde{\mathbf{E}}_n) + (1-\alpha)V_n(\mathbf{E}_n), \quad 0 < \alpha < 1 \tag{53}$$

$$[\mathbf{S}_n(\widetilde{\mathbf{E}}_n) - \mathbf{S}_n(\mathbf{E}_n)] : (\widetilde{\mathbf{E}}_n - \mathbf{E}_n) > 0 \tag{54}$$

$$(\widetilde{\mathbf{E}}_n - \mathbf{E}_n) : \ \mathbb{S}_n(\widetilde{\mathbf{E}}_n - \mathbf{E}_n) > 0 \tag{55}$$

Remark. Although the *nominal* energy density W (as a function of the transformation gradient $\mathbf{F}$, to be defined in (59)) cannot be assumed to be convex over $\mathscr{PL}in = \{\mathbf{F}, \det \mathbf{F} > 0\}$ (for formulation objectivity and solution multiplicity reasons to be discussed later), there is no objection to requiring the *metric* energy density V_n to be convex over $\mathscr{S}ym$.

Due to strict convexity, the *inverse* stress–strain linear law $\mathbf{E}_n \equiv \mathbf{S}_n^{-1}$ exists and derives from a quadratic elastic *complementary* energy density V_n^* and possesses a constant gradient called the *compliance* elasticity tensor $\mathbb{E}_n = \mathbb{S}_n^{-1}$

$$V_n^*(\mathbf{S}_n) = \frac{1}{2}\mathbf{S}_n : \mathbb{E}_n\mathbf{S}_n, \qquad V_n^*(\mathbf{O}) = 0 \tag{56}$$

$$\mathbf{E}_n(\mathbf{S}_n) = \nabla_{\mathbf{S}_n} V_n^*(\mathbf{S}_n) = \mathbb{E}_n\mathbf{S}_n, \quad \mathbf{E}_n(\mathbf{O}) = \mathbf{O} \tag{57}$$

$$\mathbb{E}_n = \mathbb{S}_n^{-1} = \nabla_{\mathbf{S}_n}\mathbf{E}_n = \nabla^2_{\mathbf{S}_n} V_n^*, \quad \mathbb{E}_n \succ 0 \tag{58}$$

The complementary energy V_n^* and the strain $\mathbf{E}_n$ are equal to zero at the stress origin. The compliance $\mathbb{E}_n = \mathbb{S}_n^{-1}$ has the same symmetries as $\mathbb{S}_n$. Moreover, V_n^* is strictly convex, $\mathbf{S}_n$ strictly monotone, and $\mathbb{E}_n$ positive definite. The direct law (50)–(52) and its inverse (56)–(58) are *objective* since $\mathbf{E}_n^* = \mathbf{E}_n$ and $\mathbf{S}_n^* = \mathbf{S}_n$, in a change of reference frame.

4.2. Nonlinear nominal law

The nonlinear elastic nominal law is obtained by substituting the definition of the metric strain $\mathbf{E}_n(\mathbf{F})$—(19) in the linear metric law $\mathbf{S}_n(\mathbf{E}_n)$—(51) and then the result into the nominal stress $\mathbf{P}(\mathbf{S}_n,\mathbf{F})$—(48).

$$W(\mathbf{F}, n) = V(\mathbf{F}^{\mathrm{T}}\mathbf{F}, n) = V_n[\mathbf{E}_n(\mathbf{F}^{\mathrm{T}}\mathbf{F})], \quad W(\mathbf{I}, n) = 0 \tag{59}$$

$$\begin{aligned}\mathbf{P}(\mathbf{F}, n) &= \nabla_{\mathbf{F}} W(\mathbf{F}, n) = \mathbf{F}\mathbf{S}(\mathbf{F}^{\mathrm{T}}\mathbf{F}, n), \quad \mathbf{P}(\mathbf{I}, n) = \mathbf{O} \\ &= \frac{2+n}{4}\mathbf{F}\mathbf{S}_n[\mathbf{E}_n(\mathbf{F}^{\mathrm{T}}\mathbf{F})] + \frac{2-n}{4}\mathbf{F}^{-\mathrm{T}}\mathbf{S}_n[\mathbf{E}_n(\mathbf{F}^{\mathrm{T}}\mathbf{F})]\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}\end{aligned} \tag{60}$$

$$\begin{aligned}\mathbb{P}(\mathbf{F}, n) &= \nabla_{\mathbf{F}}\mathbf{P}(\mathbf{F}, n), \quad \mathbb{P}(\mathbf{I}, n) = \mathbb{S}_n \\ &= \mathbf{I} \underline{\otimes} \mathbf{S}(\mathbf{F}^{\mathrm{T}}\mathbf{F}, n) + [\mathbf{F} \underline{\otimes} \mathbf{I}]\mathbb{S}(\mathbf{F}^{\mathrm{T}}\mathbf{F}, n)[\mathbf{F} \underline{\otimes} \mathbf{I}]^{\mathrm{T}} \\ &= \left[\frac{2+n}{4}\mathbf{F}\underline{\otimes}\mathbf{I} + \frac{2-n}{4}\mathbf{F}^{-\mathrm{T}} \underline{\otimes} (\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}})\right]\mathbb{S}_n\left[\frac{2+n}{4}\mathbf{F}^{\mathrm{T}} \underline{\otimes} \mathbf{I} + \frac{2-n}{4}\mathbf{F}^{-1} \underline{\otimes} (\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}})\right] \\ &\quad + \frac{2+n}{4}\mathbf{I} \underline{\otimes} \mathbf{S}_n - \frac{2-n}{4}\left[\mathbf{F}^{-\mathrm{T}} \overline{\otimes} (\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}\mathbf{S}_n\mathbf{F}^{-1})\right. \\ &\quad \left. + (\mathbf{F}^{-\mathrm{T}}\mathbf{S}_n\mathbf{F}^{-1}) \underline{\otimes} (\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}) + (\mathbf{F}^{-\mathrm{T}}\mathbf{S}_n\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}) \overline{\otimes} \mathbf{F}^{-1}\right]\end{aligned} \tag{61}$$

where $-2 \leqslant n \leqslant 2$ and the abbreviation $\mathbf{S}_n = \mathbf{S}_n[\mathbf{E}_n(\mathbf{F}^{\mathrm{T}}\mathbf{F})]$ is used for conciseness. In (48) and (60), the same letter $\mathbf{P}$ is abusively used for denoting a component and a composite stress functions with equal values $\mathbf{P}(\mathbf{S}_n, \mathbf{F}) = \mathbf{P}(\mathbf{F}, n)$.

Again, the classical StVenant–Kirchhoff law is recovered for $n = 2$; the quasilogarithmic law ($n = 0$) was suggested in Curnier and Rakotomanana (1984), Curnier and Rakotomanana (1991) and proved operational in Pietrzak (1997) and Pietrzak and Curnier, 1999. The quasilinear law is new.

It can be checked that the nominal-metric law is *objective* in the nominal sense that $\forall \mathbf{F}(\det \mathbf{F} > 0)$, $\forall \mathbf{R} = \mathbf{R}^{-\mathrm{T}}$,

$$\begin{aligned} &W(\mathbf{RF}, n) = W(\mathbf{F}, n) \\ &\mathbf{P}(\mathbf{RF}, n) = \mathbf{RP}(\mathbf{F}, n) \\ &\mathbb{P}(\mathbf{RF}, n) = [\mathbf{R} \underline{\otimes} \mathbf{I}]\mathbb{P}(\mathbf{F}, n)[\mathbf{R} \underline{\otimes} \mathbf{I}]^{\mathrm{T}} \end{aligned} \tag{62}$$

A nominal law is *isotropic* when in addition

$$\begin{aligned} &W(\mathbf{FR}^{\mathrm{T}}, n) = W(\mathbf{F}, n) \\ &\mathbf{P}(\mathbf{FR}^{\mathrm{T}}, n) = \mathbf{P}(\mathbf{F}, n)\mathbf{R}^{\mathrm{T}} \\ &\mathbb{P}(\mathbf{FR}^{\mathrm{T}}, n) = [\mathbf{I} \underline{\otimes} \mathbf{R}]\mathbb{P}(\mathbf{F}, n)[\mathbf{I} \underline{\otimes} \mathbf{R}]^{\mathrm{T}} \end{aligned} \tag{63}$$

As specified, the nominal law (59)–(61) is *consistent* about the original-natural form Ω (where $\mathbf{F} = \mathbf{I}$). Note that $\mathbb{P}(\mathbf{I}, n)$ is singular; its rank is 6 instead of 9 (the default being due to rotational freedom). If the elastic solid undergoes a rotation from its original form Ω, then objectivity (62) implies in particular that the nominal energy and stress remain equal to zero whereas the stiffness becomes equal to the rotated original material stiffness, i.e. $\forall \mathbf{R} = \mathbf{R}^{-\mathrm{T}}$,

$$\begin{aligned} &W(\mathbf{R}, n) = W(\mathbf{I}, n) = 0 \\ &\mathbf{P}(\mathbf{R}, n) = \mathbf{RP}(\mathbf{I}, n) = \mathbf{O} \\ &\mathbb{P}(\mathbf{R}, n) = [\mathbf{R} \underline{\otimes} \mathbf{I}]\mathbb{P}(\mathbf{I}, n)[\mathbf{R} \underline{\otimes} \mathbf{I}]^{\mathrm{T}} \end{aligned} \tag{64}$$

As a consequence, W *cannot* be strictly convex, $\mathbf{P}$ strictly monotone and $\mathbb{P}$ strictly positive (to see that, apply the definitions (53)–(55) to them between $\mathbf{I}$ and $\mathbf{R}$ to run into contradictions). At best, they can only be so over the subspace $\mathscr{S}ym_+$ of irrotational symmetric positive definite transformations.

4.3. Nonlinear isotropic spectral energy

In the *isotropic* case (indicated by a $\mathrel{\mathring{=}}$ sign), the elastic energy density can be expressed as a symmetric (i.e. invariant under pairwise and cyclic permutations) function Φ of the singular values $\upsilon_a = \sqrt{\gamma_a}$ of $\mathbf{F} = \mathbf{RU}$ (which are the eigenvalues of $\mathbf{U} = \sqrt{\mathbf{C}}$), called the *spectral energy* density; moreover, the first partial derivatives $\Phi_{,a}$ of Φ with respect to υ_a are the *principal stresses* π_a along the principal directions of $\mathbf{C}$ of the (Biot–) Ziegler symmetric *linear stress* tensor (Truesdell and Noll, 1965)

$$W(\mathbf{F}, n) \mathrel{\mathring{=}} \Phi(\upsilon_1, \upsilon_2, \upsilon_3; n) \tag{65}$$

$$\underline{\mathbf{S}}_1 = \mathbf{R}^{\mathrm{T}}\mathbf{P} \mathrel{\mathring{=}} \pi_a \mathbf{c}_a \otimes \mathbf{c}_a, \quad \pi_a(\upsilon_1, \upsilon_2, \upsilon_3; n) = \frac{\partial \Phi}{\partial \upsilon_a}(\upsilon_1, \upsilon_2, \upsilon_3; n) \tag{66}$$

For the isotropic metric law deriving from the elastic energy defined by

$$W(\mathbf{F}, n) \mathrel{\mathring{=}} \frac{\lambda}{2}\mathrm{tr}^2[\mathbf{E}_n(\mathbf{F}^{\mathrm{T}}\mathbf{F})] + \mu \mathrm{tr}[\mathbf{E}_n^2(\mathbf{F}^{\mathrm{T}}\mathbf{F})] \tag{67}$$

the spectral energy density Φ, the principal stresses $\pi_a = \Phi_{,a}$ and the (symmetric) spectral stiffness $\pi_{a,b} = \Phi_{,ab} = \Phi_{,ba}$ are equal to

$$\Phi(\upsilon_c; n) \equiv \Phi(\upsilon_1, \upsilon_2, \upsilon_3; n)$$

$$= \frac{\lambda}{2}\left[e_n(\upsilon_1^2) + e_n(\upsilon_2^2) + e_n(\upsilon_3^2)\right]^2 + \mu[e_n^2(\upsilon_1^2) + e_n^2(\upsilon_2^2) + e_n^2(\upsilon_3^2)] \tag{68}$$

$$\pi_a(\upsilon_c; n) = \Phi_{,a}(\upsilon_c; n) \equiv \frac{\partial \Phi}{\partial \upsilon_a}(\upsilon_1, \upsilon_2, \upsilon_3; n)$$

$$= \left\{\lambda[e_n(\upsilon_1^2) + e_n(\upsilon_2^2) + e_n(\upsilon_3^2)] + 2\mu e_n(\upsilon_a^2)\right\} 2\upsilon_a e_n'(\upsilon_a^2) \tag{69}$$

$$\pi_{a,b}(\upsilon_c; n) = \Phi_{,ab}(\upsilon_c; n) \equiv \frac{\partial^2 \Phi}{\partial \upsilon_a \partial \upsilon_b}(\upsilon_1, \upsilon_2, \upsilon_3; n)$$

$$= (\lambda + 2\mu\delta_{ab}) 4\upsilon_a e_n'(\upsilon_a^2)\upsilon_b e_n'(\upsilon_b^2)$$

$$+ \{\lambda[e_n(\upsilon_1^2) + e_n(\upsilon_2^2) + e_n(\upsilon_3^2)] + 2\mu e_n(\upsilon_a^2)\}[2e_n'(\upsilon_b^2) + 4\upsilon_b e_n''(\upsilon_b^2)]\delta_{ab} \tag{70}$$

where e_n is the metric scale defined in (17) and $e_n' = \mathrm{d}e_n/\mathrm{d}\upsilon^2 (\neq \mathrm{d}e_n/\mathrm{d}\upsilon)$ its derivative with respect to $\gamma = \upsilon^2$ derived in (18), respectively equal to

$$e_n(\upsilon^2) = \frac{2+n}{8}\upsilon^2 - \frac{n}{4} - \frac{2-n}{8}\upsilon^{-2}, \quad e_n'(\upsilon^2) = \frac{2+n}{8} + \frac{2-n}{8}\upsilon^{-4}$$

The consistency conditions at the origin take the form

$$\Phi(1,1,1;n) = 0, \quad \Phi_{,a}(1,1,1;n) = 0, \quad \Phi_{,ab}(1,1,1;n) = \lambda + 2\mu\delta_{ab}$$

Note that the Hessian matrix at the origin coincides with the usual spectral stiffness of linear elasticity (as it must)

$$[\Phi_{,ab}(1,1,1;n)] = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda \\ \lambda & \lambda + 2\mu & \lambda \\ \lambda & \lambda & \lambda + 2\mu \end{bmatrix}$$

The spectral form Φ is useful for assessing the convexity properties of W, as discussed in the next paragraph, because the υ_a are homogeneous (of degree-1) functions of **U** and **F**. The following finite quotients $\delta\Phi_{ab}$ and $\Delta\Phi_{ab}$ will also be useful for this matter

$$\delta\Phi_{ab}(\upsilon_c; n) \equiv \frac{\Phi_{,a} - \Phi_{,b}}{\upsilon_a - \upsilon_b}(\upsilon_1, \upsilon_2, \upsilon_3; n)$$

$$= \lambda\left[e_n(\upsilon_1^2) + e_n(\upsilon_2^2) + e_n(\upsilon_3^2)\right] 2\frac{\upsilon_a e_n'(\upsilon_a^2) - \upsilon_b e_n'(\upsilon_b^2)}{\upsilon_a - \upsilon_b}$$

$$+ 4\mu\frac{\upsilon_a e_n(\upsilon_a^2)e_n'(\upsilon_a^2) - \upsilon_b e_n(\upsilon_b^2)e_n'(\upsilon_b^2)}{\upsilon_a - \upsilon_b} \tag{71}$$

$$\Delta\Phi_{ab}(\upsilon_c; n) \equiv \frac{\upsilon_a\Phi_{,a} - \upsilon_b\Phi_{,b}}{\upsilon_a - \upsilon_b}(\upsilon_1, \upsilon_2, \upsilon_3; n)$$

$$= \lambda\left[e_n(\upsilon_1^2) + e_n(\upsilon_2^2) + e_n(\upsilon_3^2)\right] 2\frac{\upsilon_a^2 e_n'(\upsilon_a^2) - \upsilon_b^2 e_n'(\upsilon_b^2)}{\upsilon_a - \upsilon_b} + 4\mu\frac{\upsilon_a^2 e_n(\upsilon_a^2)e_n'(\upsilon_a^2) - \upsilon_b^2 e_n(\upsilon_b^2)e_n'(\upsilon_b^2)}{\upsilon_a - \upsilon_b} \tag{72}$$

5. Metric convexity

In this section, the *monotonicity* of the isotropic nominal metric law is examined in order to delimit its range of applicability, meaning the range of strains over which existence of a solution can be ensured (for different values of n and of the relevant elastic constant ν). Appropriate background for this topic can be found in the books on mathematical elasticity or inelasticity by Truesdell and Noll (1965), Marsden and Hughes (1983), Ciarlet (1988), Silhavy (1997) and on the calculus of variations by Dacorogna (1988).

5.1. Question of existence of a solution

The problem of metric nonlinear elasticity consists in solving the equilibrium equation $\mathrm{Div}\,\mathbf{P} = \mathbf{g}$ (where $\mathbf{g}$ is a volume force density) together with the nominal law (60), subjected to suitable boundary conditions. General conditions for the *existence* of a solution to this problem are difficult to establish. Note that *uniqueness* is not the rule in large transformations, as illustrated by buckling phenomena, for instance. A powerful approach for addressing the existence issue is the direct method of the calculus of variations, which consists in showing the existence of a minimizer of the total energy of the loaded elastic solid, based on the *nominal* elastic energy density $W: \mathbf{F} \mapsto W(\mathbf{F})$, under appropriate relaxed convexity conditions, cf. e.g. (Dacorogna, 1988). An optimal *necessary and sufficient* condition (NSC) for existence of a solution is the *quasiconvexity* of the total energy involving $\int_\Omega W\,\mathrm{d}V$, discovered by Morrey (1952). Unfortunately, quasiconvexity is very difficult to interpret and verify, because it is a global requirement over Ω. A simpler *necessary* condition for existence is the *ellipticity* or *rank-one convexity* of W (monotonicity of $\mathbf{P}$, positivity of $\mathbb{P}$), initiated by Legendre and confirmed by Hadamard, cf. e.g. (Truesdell and Noll, 1965; Ball, 1977a,b; Dacorogna, 1988); obtained by restricting trial transformation gradients in the definition of rank-three convexity to pairs differing by a rank-1 modification

$$\forall \mathbf{F}, \widetilde{\mathbf{F}} = \mathbf{F} + \mathbf{f} \otimes \mathbf{g} \in \mathscr{L}in \qquad \text{rank-one} \tag{73}$$

$$W[\alpha\widetilde{\mathbf{F}} + (1-\alpha)\mathbf{F}] \leqslant \alpha W(\widetilde{\mathbf{F}}) + (1-\alpha)W(\mathbf{F})(0 \leqslant \alpha \leqslant 1) \qquad \text{convexity} \tag{74}$$

$$[\mathbf{P}(\widetilde{\mathbf{F}}) - \mathbf{P}(\mathbf{F})] : (\widetilde{\mathbf{F}} - \mathbf{F}) \geqslant 0 \qquad \text{monotonicity} \tag{75}$$

$$(\widetilde{\mathbf{F}} - \mathbf{F}) : \ \mathbb{P}(\mathbf{F})(\widetilde{\mathbf{F}} - \mathbf{F}) \geqslant 0 \qquad \text{positivity} \tag{76}$$

Rank-one convexity can be interpreted as a *directional* convexity, especially when starting from $\mathbf{F} = \mathbf{I}$. Note that $\det \widetilde{\mathbf{F}} = \det \mathbf{F}(1 + \mathbf{g} \cdot \mathbf{F}^{-1}\mathbf{f}) > 0$ requires $\mathbf{g} \cdot \mathbf{F}^{-1}\mathbf{f} > -1$ for orientation consistency. Note also that, due to objectivity, (73) can be reduced to $\forall \mathbf{F} = \mathbf{U}, \widetilde{\mathbf{F}} = \mathbf{U} + \mathbf{h} \otimes \mathbf{g}$, but $\widetilde{\mathbf{F}}^{\mathrm{T}} \neq \widetilde{\mathbf{F}}$ in general.

5.2. Characterisation of rank-one convexity for an isotropic material

When the material is *isotropic*, the elastic energy can be written in spectral form as a symmetric function Φ of the principal stretches as in (65)

$$W(\mathbf{F}) = W(\mathbf{RFR}^{\mathrm{T}}) \mathrel{\hat{=}} \Phi(\upsilon_1, \upsilon_2, \upsilon_3) \equiv \Phi(\boldsymbol{\upsilon}) = \Phi[\boldsymbol{\upsilon}(\mathbf{F})]$$

$$\Phi(\upsilon_1, \upsilon_2, \upsilon_3) = \Phi(\upsilon_2, \upsilon_3, \upsilon_1) = \Phi(\upsilon_3, \upsilon_1, \upsilon_2) = \Phi(\upsilon_2, \upsilon_1, \upsilon_3)$$

The nominal isotropic energy W is *rank-1 convex* if and only if the spectral energy gradient $\mathbf{\nabla}_v\Phi = (\Phi_{,a})$ satisfies the *Baker–Ericksen* inequalities and its modified Hessians (Hadeler, 1983; Simpson and Spector, 1983; Dacorogna, 1988; Rosakis and Simpson, 1995; Silhavy, 1999; Dacorogna, 2001),

$$\mathbf{\nabla}_v^2\Phi_{+++} \equiv \begin{bmatrix} \Phi_{,11} & \Phi^+_{,12} & \Phi^+_{,13} \\ - & \Phi_{,22} & \Phi^+_{,23} \\ \text{Sym.} & - & \Phi_{,33} \end{bmatrix} \quad \text{and} \quad \mathbf{\nabla}_v^2\Phi_{+--} \equiv \begin{bmatrix} \Phi_{,11} & \Phi^-_{,12} & \Phi^-_{,13} \\ - & \Phi_{,22} & \Phi^+_{,23} \\ \text{Sym.} & - & \Phi_{,33} \end{bmatrix}$$

where $\Phi^+_{,12} \equiv \Phi_{,12} + \frac{\Phi_{,1}-\Phi_{,2}}{v_1-v_2}$ *and* $\Phi^-_{,12} \equiv -\Phi_{,12} + \frac{\Phi_{,1}+\Phi_{,2}}{v_1+v_2}$ are *copositive* (cf. (Simpson and Spector, 1983; Silhavy, 1999) for the original definition), which hold if and only if the spectral derivatives or finite quotients satisfy the inequalities

$$\Phi_{,11} \geqslant 0, \quad \Delta\Phi_{12} \equiv \frac{v_1\Phi_{,1} - v_2\Phi_{,2}}{v_1 - v_2} \geqslant 0 \quad (v_1 \neq v_2) \tag{77}$$

$$\text{pm}\Phi^+_{12} \equiv \sqrt{\Phi_{,11}\Phi_{,22}} + \Phi^+_{,12} \geqslant 0 (v_1 \neq v_2), \quad \text{pm}\Phi^-_{12} \equiv \sqrt{\Phi_{,11}\Phi_{,22}} + \Phi^-_{,12} \geqslant 0$$

either

$$\delta\text{et}\mathbf{\nabla}_v^2\Phi_{+++} \equiv \sqrt{\Phi_{,11}\Phi_{,22}\Phi_{,33}} + \sqrt{\Phi_{,11}}\Phi^+_{,23} + \sqrt{\Phi_{,22}}\Phi^+_{,31} + \sqrt{\Phi_{,33}}\Phi^+_{,12} \geqslant 0$$
$$\delta\text{et}\mathbf{\nabla}_v^2\Phi_{+--} \equiv \sqrt{\Phi_{,11}\Phi_{,22}\Phi_{,33}} + \sqrt{\Phi_{,11}}\Phi^+_{,23} + \sqrt{\Phi_{,22}}\Phi^-_{,31} + \sqrt{\Phi_{,33}}\Phi^-_{,12} \geqslant 0$$

or

$$\det \mathbf{\nabla}_v^2\Phi_{+++} = \Phi_{,11}\Phi_{,22}\Phi_{,33} + 2\Phi^+_{,23}\Phi^+_{,31}\Phi^+_{,12} - \Phi_{,11}\Phi^{+2}_{,23} - \Phi_{,22}\Phi^{+2}_{,31} - \Phi_{,33}\Phi^{+2}_{,12} \geqslant 0$$
$$\det \mathbf{\nabla}_v^2\Phi_{+--} = \Phi_{,11}\Phi_{,22}\Phi_{,33} + 2\Phi^+_{,23}\Phi^-_{,31}\Phi^-_{,12} - \Phi_{,11}\Phi^{+2}_{,23} - \Phi_{,22}\Phi^{-2}_{,31} - \Phi_{,33}\Phi^{-2}_{,12} \geqslant 0$$

At the start, there are four modified Hessians $\mathbf{\nabla}_v^2\Phi_{+++}$, $\mathbf{\nabla}_v^2\Phi_{+--}$, $\mathbf{\nabla}_v^2\Phi_{-+-}$ and $\mathbf{\nabla}_v^2\Phi_{--+}$ which should be copositive in (77), but due to the permutation and cyclic symmetries, the last two are equivalent to the second one. The condition $\Delta\Phi_{12} \geqslant 0$ is equivalent to the *Baker–Ericksen* (B–E) inequalities (Truesdell and Noll, 1965).

Remark. The inequalities (77) are better understood after applying them to the linear isotropic Cauchy–Biot law $\underline{\mathbf{S}}_1(\underline{\mathbf{E}}_1) = \lambda(\text{tr}\underline{\mathbf{E}}_1)\mathbf{I} + 2\mu\underline{\mathbf{E}}_1$, between the linear strain $\underline{\mathbf{E}}_1 = \mathbf{U} - \mathbf{I}$ and the conjugate rotated stress $\underline{\mathbf{S}}_1 = \mathbf{R}^{\mathrm{T}}\mathbf{P}$. It can be shown that, under the usual provisions on the elastic constants, this archetype objective linear law is not rank-one convex, as discovered by (Ball, 1984).

5.3. Rank-one convexity domains of the metric law

Given the explicit expression (68) of $\Phi(v_1, v_2, v_3; n, \nu)$, *rank-one* convexity was tested numerically with the commercial code *Mathematica*. The results for plane, axisymmetric and three-dimensional strains are summarized in Figs. 2 and 3. It is checked that the extreme laws $n = \pm 2$ lead to violation of all convexity conditions in tension ($n = -2$) and in hydrostatic pressure ($n = 2$). The importance of Poisson's ratio effect on the extent of the convexity regions is illustrated in Fig. 3. The intermediate laws $n = 0, 1$ demonstrate an extended area of rank-one convexity around the original state of strain. However, the situation deteriorates when $\nu \to 1/2$. Clearly, the classical experiments (dotted lines) are not sufficient for ensuring the overall rank-one convexity condition necessary for existence of a solution to all three-dimensional boundary value problems. Finally, it should be emphasized that the quasilogarithmic and quasilinear laws ($n = 0, 1$) are stable over substantially larger domains around the original state of strain than the quadratic and quadhyperbolic ones ($n = \pm 2$).

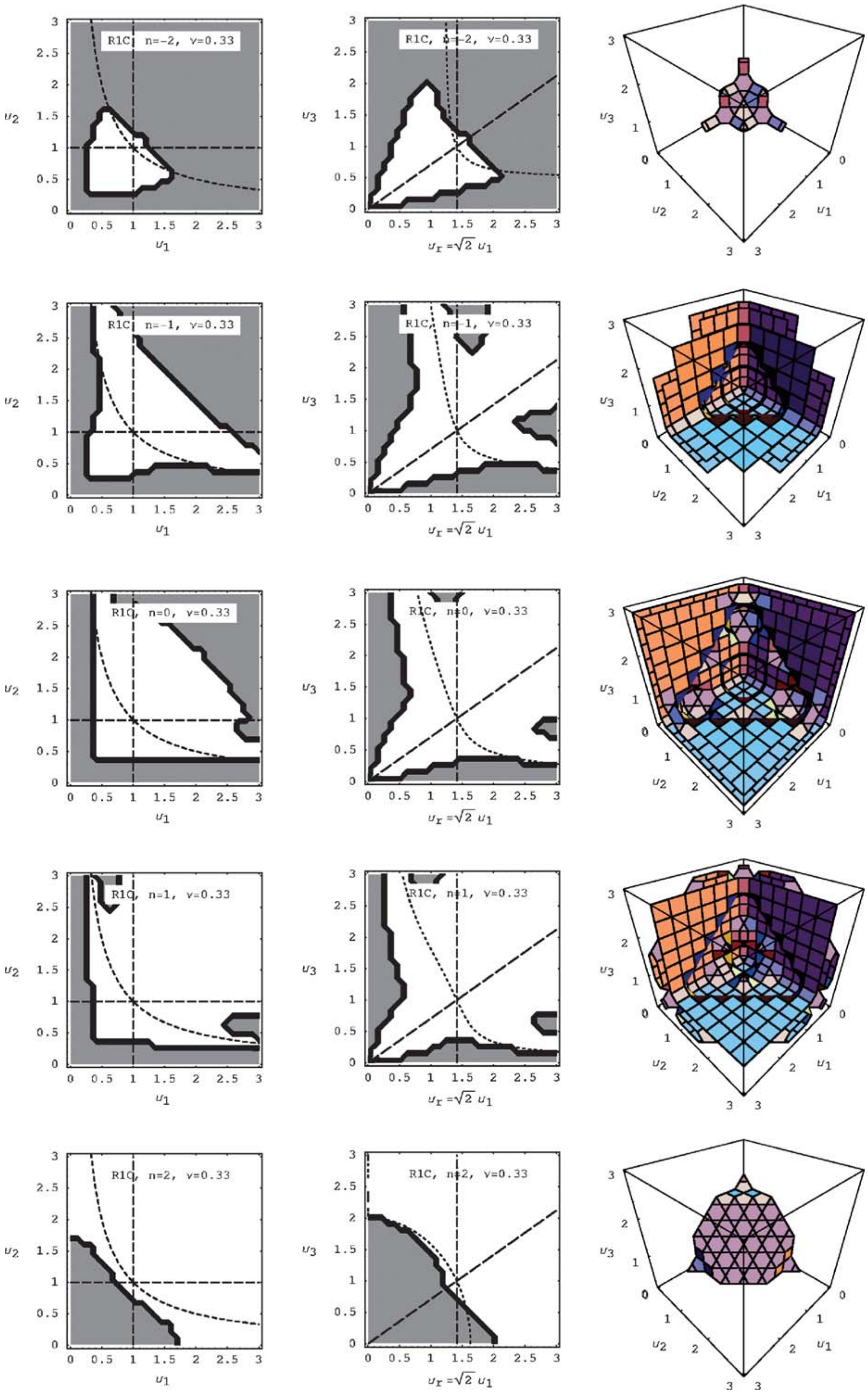

Fig. 2. Regions of the stretch eigenvalue space where rank-one convexity is violated (in grey) in plane strain (left column), axisymmetric strain (center column) and triaxial strain (right column) for $n = -2, -1, 0, 1, 2$ (rows) and $\nu = 1/3$. In the 2D plots, the dotted lines are the deformation paths of elongation and pure glide in plain strain (left column) and dilatation, elongation and traction in axisymmetric strain (center column). In the 3D plots (right column), the surfaces correspond to the boundaries between the white and grey regions.

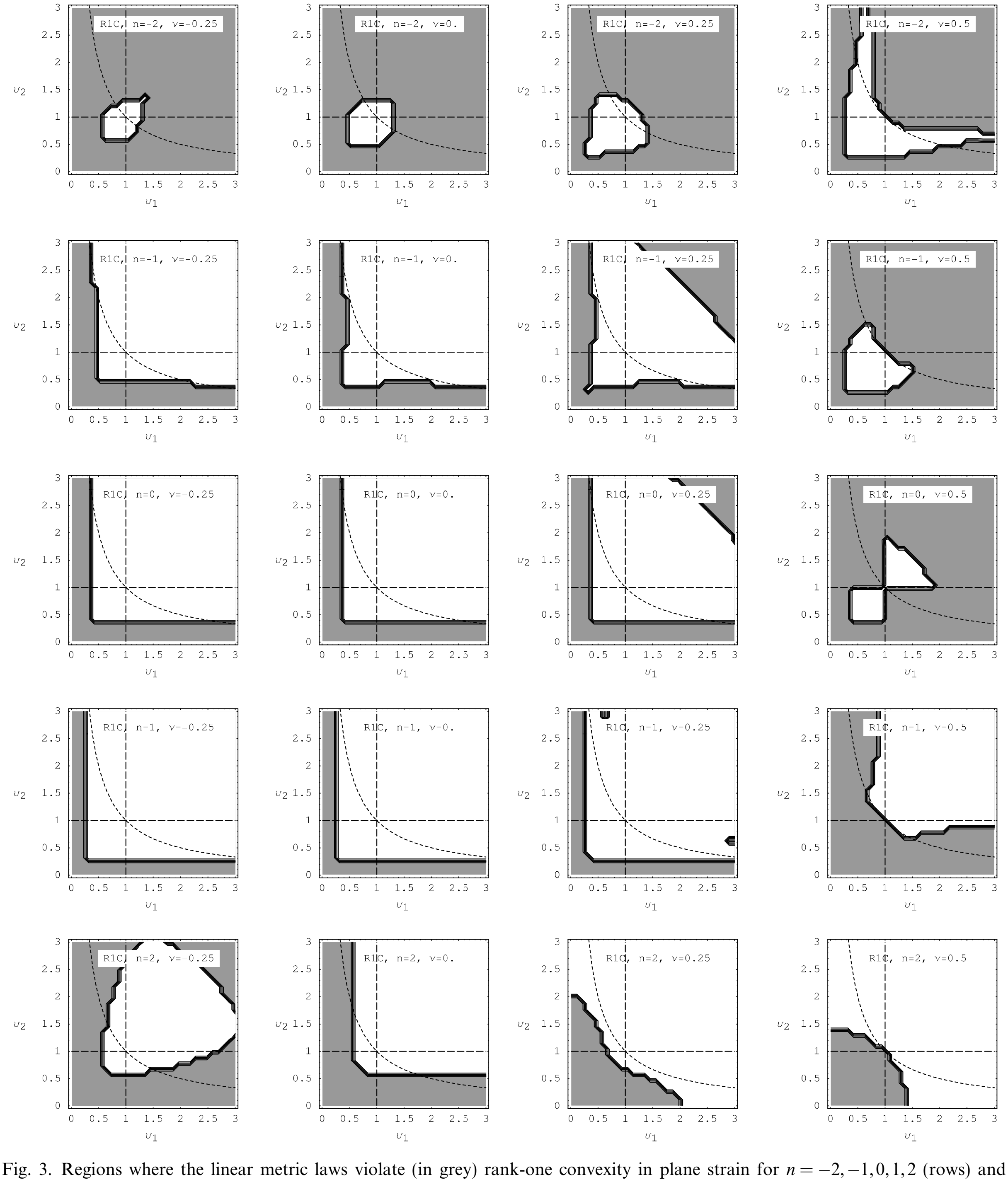

Fig. 3. Regions where the linear metric laws violate (in grey) rank-one convexity in plane strain for $n = -2, -1, 0, 1, 2$ (rows) and $\nu = -0.25, 0, 0.25, 0.5$ (columns). The dotted lines are the paths of elongation and pure glide.

6. Homogeneous strain–stress illustrations

The family of isotropic elastic metric strain–stress laws will now be illustrated by means of classical homogeneous stress–strain states, many of which correspond to standard rheological experiments on isotropic materials, namely:

- *dilatation*, i.e. spherical tension–dilatation (or pressure–concentration), as in a ball under pressure,
- *simple elongation*, i.e. tritraction–unielongation (or contraction-shortening),
- *simple traction*, i.e. unitraction–trielongation (or contraction-shortening), as in a rod under traction,
- *pure glide*, i.e. tritraction–reciprocal-bielongation,
- *pure shear*, i.e. opposite-bitraction–trielongation, as in a thin tube in torsion.

All five are symmetric *tension-stretch* states defined as follows.

6.1. Pure tension-stretch

A *pure stretch* homogeneous symmetric deformation $\mathbf{F} = \mathbf{U} = \mathbf{U}^{\mathrm{T}}(\mathbf{R} = \mathbf{I})$ in its spectral form is

$$\mathbf{F} = \upsilon_a \mathbf{c}_a \otimes \mathbf{c}_a, \quad 0 < \upsilon_a < \infty, \quad \mathbf{c}_a \cdot \mathbf{c}_b = \delta_{ab}, \quad a, b = 1, 3 \tag{78}$$

The corresponding coaxial *pure nominal tension* $\mathbf{P} = \underline{\mathbf{S}}_1 = \underline{\mathbf{S}}_1^{\mathrm{T}}$ is

$$\begin{aligned} \mathbf{P}(n) &= \pi_a(\upsilon_c; n)\mathbf{c}_a \otimes \mathbf{c}_a \\ \pi_a(\upsilon_c; n) &= \left\{\lambda[e_n(\upsilon_1^2) + e_n(\upsilon_2^2) + e_n(\upsilon_3^2)] + 2\mu e_n(\upsilon_a^2)\right\} 2\upsilon_a e_n'(\upsilon_a^2) \\ &= \left\{\lambda\left[\frac{2+n}{8}(\upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2) - \frac{3n}{4} - \frac{2-n}{8}(\upsilon_1^{-2} + \upsilon_2^{-2} + \upsilon_3^{-2})\right]\right. \\ &\quad \left. + 2\mu\left[\frac{2+n}{8}\upsilon_a^2 - \frac{n}{4} - \frac{2-n}{8}\upsilon_a^{-2}\right]\right\}\left(\frac{2+n}{4}\upsilon_a + \frac{2-n}{4}\upsilon_a^{-3}\right) \end{aligned} \tag{79}$$

since once again $e_n(\upsilon^2) \equiv \frac{2+n}{8}\upsilon^2 - \frac{n}{4} - \frac{2-n}{8}\upsilon^{-2}$ and $e_n'(\upsilon^2) = \frac{2+n}{8} + \frac{2-n}{8}\upsilon^{-4}$.

The matrices of $\mathbf{F} = \mathbf{U}$ and $\mathbf{P} = \underline{\mathbf{S}}_1$ in the principal basis $\mathbf{c}_a$ are

$$[\mathbf{F}] = \begin{bmatrix} \upsilon_1 & 0 & 0 \\ 0 & \upsilon_2 & 0 \\ 0 & 0 & \upsilon_3 \end{bmatrix}, \quad [\mathbf{P}(n)] = \begin{bmatrix} \pi_1(\upsilon_c; n) & 0 & 0 \\ 0 & \pi_2(\upsilon_c; n) & 0 \\ 0 & 0 & \pi_3(\upsilon_c; n) \end{bmatrix}$$

6.2. Dilatation

A spherical *dilatation* (or concentration) is characterized by a radial stretch $\upsilon_1 = \upsilon_2 = \upsilon_3 = \upsilon$

$$\mathbf{F} = \upsilon \mathbf{I}, \quad 0 < \upsilon < \infty \tag{80}$$

The nominal stress is a spherical *tension* (or pressure) characterized by a radial component $\pi_1 = \pi_2 = \pi_3 = \pi$

$$\mathbf{P}(n) = \pi(\upsilon; n)\mathbf{I}; \quad \pi(\upsilon; n) = (3\lambda + 2\mu)e_n(\upsilon^2)2\upsilon e_n'(\upsilon^2) = 3\kappa f_n(\upsilon) \tag{81}$$

where f_n is the *metric nominal elongation scale function* (measuring the radial stretch) defined by

$$\begin{aligned} f_n(\upsilon) &\equiv 2\upsilon e_n'(\upsilon^2)e_n(\upsilon^2), \quad f_n(1) = 0 \\ &= \frac{1}{2}\left(\frac{2+n}{4}\right)^2\upsilon^3 - \frac{n}{4}\frac{2+n}{4}\upsilon - \frac{n}{4}\frac{2-n}{4}\upsilon^{-3} - \frac{1}{2}\left(\frac{2-n}{4}\right)^2\upsilon^{-5} \end{aligned} \tag{82}$$

The matrices of $\mathbf{F}$ and $\mathbf{P}$ in any orthonormal basis are

$$[\mathbf{F}] = \begin{bmatrix} \upsilon & 0 & 0 \\ 0 & \upsilon & 0 \\ 0 & 0 & \upsilon \end{bmatrix}, \quad [\mathbf{P}(n)] = \begin{bmatrix} \pi(\upsilon;n) & 0 & 0 \\ 0 & \pi(\upsilon;n) & 0 \\ 0 & 0 & \pi(\upsilon;n) \end{bmatrix}$$

Therefore, the nominal scale f_n defined in (82) governs the tension–dilatation response. This calls for its analysis. The first and second derivatives of f_n with respect to υ are

$$f_n'(\upsilon) = \frac{3}{2}\left(\frac{2+n}{4}\right)^2 \upsilon^2 - \frac{n}{4}\frac{2+n}{4} + 3\frac{n}{4}\frac{2-n}{4}\upsilon^{-4} + \frac{5}{2}\left(\frac{2-n}{4}\right)^2 \upsilon^{-6}, \quad f_n'(1) = 1 \tag{83}$$

$$f_n''(\upsilon) = 3\left(\frac{2+n}{4}\right)^2 \upsilon - 12\frac{n}{4}\frac{2-n}{4}\upsilon^{-5} - 15\left(\frac{2-n}{4}\right)^2 \upsilon^{-7}, \quad f_n''(1) = 3(n-1)$$

It can be shown that the nominal scale function f_n is *monotone* (i.e. that $f_n'(\upsilon) > 0\ \forall \upsilon \in \mathscr{R}_+$) for all n within the range $-1.703 \leqslant n \leqslant 1.979$. For $n = 1$, the inflexion point is at the unit stretch since $f_1''(1) = 0$. It is interesting to locate the stretches $\upsilon_n' \equiv f_n'^{-1}(0)$ and $\upsilon_n'' \equiv f_n''^{-1}(0)$ where the nominal scale f_n, and hence the stress–strain law $\pi(\upsilon;n)$, reaches a minimum ($n > 1.979$) or a maximum ($n < -1.703$) and where it has its (unique) inflexion point, respectively. For the most representative values of n, they are

n	2	1.979	1	0	−1	−1.703	−2
υ_n'	$\frac{\sqrt{3}}{3} \approx 0.58$	0.472	–	–	–	1.602	$\sqrt{\frac{5}{3}} \approx 1.29$
υ_n''	0	0.472	1	$\sqrt[8]{5} \approx 1.22$	1.45	1.602	$\sqrt{\frac{5}{2}} \approx 1.58$

Taking $3\kappa = 1$ for simplicity, the graphs of the nominal metric laws $\pi(\upsilon;n) = f_n(\upsilon)$ are plotted in Fig. 4 for the integer and critical values of n.

6.3. Pressure–volume supplement

In tension–dilatation, common sense suggests that the *spatial pressure* τ (i.e. the principal value of the spatial stress $\mathbf{T}$, negative in compression and positive in tension) should be a monotone function of the *volume change* $J = \det \mathbf{F} = \upsilon^3$

$$\tau : J \mapsto \tau(J) \left| \frac{d\tau}{dJ} > 0 \quad (\forall J \in \mathscr{R}_+) \right. \tag{84}$$

This condition is mechanically reasonable because it requires the state law of a perfect elastic fluid, for which the energy W depends on $\mathbf{F}$ through its determinant J alone, to be in agreement with basic experiments. It is also mathematically reasonable because it can be shown that the energy W is polyconvex with respect to J if and only if $\frac{d\tau}{dJ} = \frac{d^2\Phi}{dJ^2} = \frac{d^2W}{dJ^2} > 0$ (Leblond, 1992).

In view of the relationship between the (Cauchy) spatial stress $\mathbf{T}$ and the nominal stress $\mathbf{P}$ (or the rotated stress $\underline{\mathbf{S}}_1$) $\mathbf{T} = J^{-1}\mathbf{P}\mathbf{F}^{\mathrm{T}} = J^{-1}\mathbf{R}\underline{\mathbf{S}}_1\mathbf{U}\mathbf{R}^{\mathrm{T}}$, the spatial pressure τ is related to the nominal one π by $\tau = \upsilon^{-2}\pi$. It follows that the spatial pressure–volume metric law is

$$\tau(J;n) = 3\kappa J^{-\frac{2}{3}} f_n(J^{\frac{1}{3}}), \quad \frac{d\tau}{dJ}(J) = \kappa J^{-\frac{5}{3}}\left[J^{\frac{1}{3}} f_n'(J^{\frac{1}{3}}) - 2 f_n(J^{\frac{1}{3}})\right] \tag{85}$$

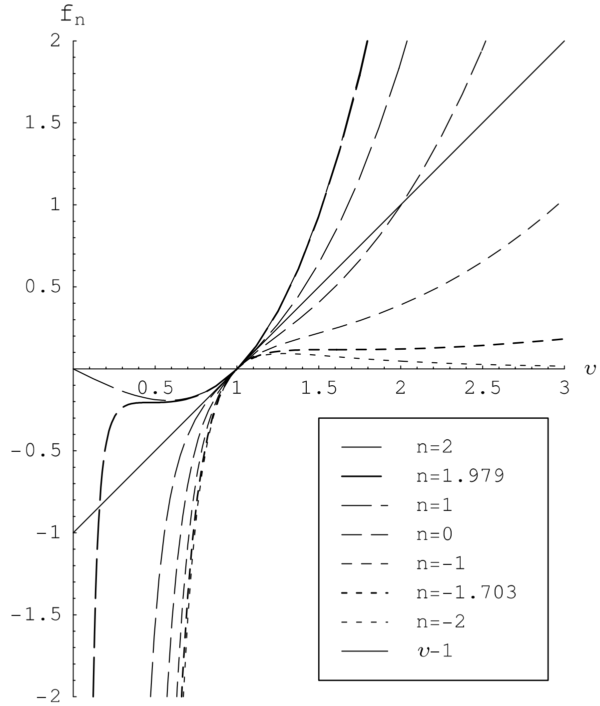

Fig. 4. Graphs $f = f_n(\upsilon)$ of the nominal scale function f_n (and hereby of the nominal radial tension–dilatation and axial traction–elongation laws, e.g. $\pi = \pi(\upsilon; n)$) for $n = -2, -1.703, -1, 0, 1, 1.979, 2$ (and relevant values of ε and ν).

Since $J > 0$, the spatial pressure–volume law is monotone if and only if the nominal scale function satisfies

$$\tilde{f}_n(\upsilon) \equiv \upsilon f_n'(\upsilon) - 2f_n(\upsilon) > 0, \quad 0 < \upsilon < \infty$$
$$= \frac{1}{2}\left(\frac{2+n}{4}\right)^2 \upsilon^3 + \frac{n}{4}\frac{2+n}{4}\upsilon + 5\frac{n}{4}\frac{2-n}{4}\upsilon^{-3} + 3\left(\frac{2-n}{4}\right)^2 \upsilon^{-5} > 0$$

On this ground, it can be shown that the *spatial pressure–volume metric laws are monotone for*: $-0.970 < n \leqslant 2$, which is another feature.

Taking $\kappa = 1$, the graphs $\tau = \tau(J; n)$ of the spatial pressure–volume laws are plotted in Fig. 5 for the integer and critical values of n.

6.4. Simple elongation

Consider now a confined uniaxial *simple elongation* (or shortening) characterized by $\upsilon_1 = \upsilon$; $\upsilon_2 = \upsilon_3 = 1$

$$\mathbf{F} = \upsilon \mathbf{c}_1 \otimes \mathbf{c}_1 + \mathbf{c}_2 \otimes \mathbf{c}_2 + \mathbf{c}_3 \otimes \mathbf{c}_3, \quad 0 < \upsilon < \infty \tag{86}$$

The nominal stress is an oval triaxial *traction* (or contraction) characterized by $\pi_1 = \pi$; $\pi_2 = \pi_3 = \pi_\perp$

$$\mathbf{P}(n) = \pi(\upsilon; n)\mathbf{c}_1 \otimes \mathbf{c}_1 + \pi_\perp(\upsilon; n)[\mathbf{c}_2 \otimes \mathbf{c}_2 + \mathbf{c}_3 \otimes \mathbf{c}_3]$$
$$\pi(\upsilon; n) = (\lambda + 2\mu)e_n(\upsilon^2)2\upsilon e_n'(\upsilon^2) = \frac{1-\nu}{(1+\nu)(1-2\nu)}\varepsilon f_n(\upsilon) \tag{87}$$
$$\pi_\perp(\upsilon; n) = \lambda e_n(\upsilon^2)$$

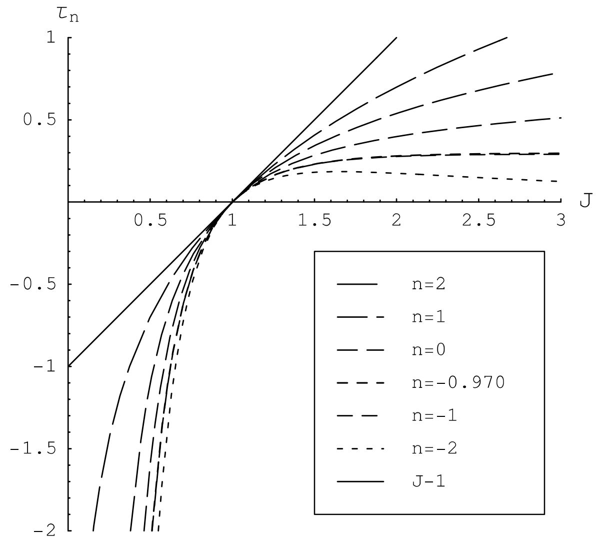

Fig. 5. Graphs $\tau = \tau(J;n)$ of the spatial pressure–volume response τ defined in (85) for $n = -2, -0.970, -1, 0, 1, 2$ (and $\kappa = 1$).

where f_n is the *same* nominal metric scale function as in tension–dilatation, defined in (82), but measuring the axial stretch (instead of the radial one) this time. Therefore, the nominal metric laws are exactly the same in simple elongation and dilatation, except for the bulk modulus $3\kappa = 3\lambda + 2\mu$ being replaced by the simple elongation one $\varepsilon^* = \frac{1-\nu}{(1+\nu)(1-2\nu)}\varepsilon = \lambda + 2\mu$.

The matrices of **F** and **P** in the orthonormal spectral basis are

$$[\mathbf{F}] = \begin{bmatrix} \upsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [\mathbf{P}(n)] = \begin{bmatrix} \pi(\upsilon;n) & 0 & 0 \\ 0 & \pi_\perp(\upsilon;n) & 0 \\ 0 & 0 & \pi_\perp(\upsilon;n) \end{bmatrix}$$

The graphs of the nominal metric traction–elongation laws $\pi(\upsilon;n) = f_n(\upsilon)$ are of course the same as in Fig. 4, provided $\varepsilon^* = 1$.

Simple elongation is instructive but hard to realize. Simple traction is easier.

6.5. Simple traction

Consider next an *oval triaxial elongation* (or shortening) characterized by $\upsilon_1 = \upsilon$; $\upsilon_2 = \upsilon_3 = \upsilon_\perp$

$$\mathbf{F} = \upsilon\mathbf{c}_1 \otimes \mathbf{c}_1 + \upsilon_\perp[\mathbf{c}_2 \otimes \mathbf{c}_2 + \mathbf{c}_3 \otimes \mathbf{c}_3], \quad 0 < \upsilon, \upsilon_\perp < \infty \tag{88}$$

and such that the nominal stress is a uniaxial *simple traction* (or contraction) characterized by $\pi_1 = \pi$; $\pi_2 = \pi_3 = \pi_\perp = 0$

$$\begin{aligned} \mathbf{P}(n) &= \pi(\upsilon, \upsilon_\perp; n)\mathbf{c}_1 \otimes \mathbf{c}_1 \\ \pi(\upsilon, \upsilon_\perp; n) &= \left[(\lambda + 2\mu)e_n(\upsilon^2) + 2\lambda e_n(\upsilon_\perp^2)\right]2\upsilon e_n'(\upsilon^2) \\ \pi_\perp(\upsilon, \upsilon_\perp; n) &= \left[2(\lambda + \mu)e_n(\upsilon_\perp^2) + \lambda e_n(\upsilon^2)\right]2\upsilon_\perp e_n'(\upsilon_\perp^2) = 0 \end{aligned} \tag{89}$$

Since $2\upsilon_\perp e_n'(\upsilon_\perp^2) > 0$, the zero transversal stress condition implies that

$$\begin{aligned} e_n(\upsilon_\perp^2) &= -\frac{\lambda}{2(\lambda+\mu)} e_n(\upsilon^2) = -\nu e_n(\upsilon^2) \\ \Longleftrightarrow \upsilon_\perp(\upsilon) &= \sqrt{e_n^{-1}[-\nu e_n(\upsilon^2)]} \end{aligned} \tag{90}$$

as expected (note that for the quasi-linear law $n = 1$, $\upsilon_\perp(\upsilon) - 1 \approx -\nu(\upsilon - 1)$).

It follows that the *axial stress-stretch law* reduces to

$$\pi(\upsilon; n) = \frac{\mu(3\lambda+\mu)}{\lambda+\mu} e_n(\upsilon^2) 2\upsilon e_n'(\upsilon^2) = \varepsilon f_n(\upsilon) \tag{91}$$

where f_n is again the *same* nominal metric scale function as in dilatation and simple elongation, defined in (82), measuring the axial stretch. Therefore, the nominal metric laws are also the same in simple traction as in simple elongation and in dilatation, except for the modulus becoming the familiar Young's modulus $\varepsilon = \frac{\mu(3\lambda+\mu)}{\lambda+\mu}$.

The matrices of **F** and **P** in the orthonormal spectral basis are

$$[\mathbf{F}] = \begin{bmatrix} \upsilon & 0 & 0 \\ 0 & \upsilon_\perp(\upsilon) & 0 \\ 0 & 0 & \upsilon_\perp(\upsilon) \end{bmatrix}, \quad [\mathbf{P}(n)] = \begin{bmatrix} \pi(\upsilon; n) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The traction–elongation graphs $\pi = \pi(\upsilon; n) = f_n(\upsilon)$ are again the same as in Fig. 4, provided $\varepsilon = 1$ this time.

6.6. Pure glide

Consider now a *pure glide*, i.e. an *isovolumic reciprocal plane stretch*, characterized by $\upsilon_1 = \upsilon$; $\upsilon_2 = 1/\upsilon$; $\upsilon_3 = 1$

$$\mathbf{F} = \upsilon \mathbf{c}_1 \otimes \mathbf{c}_1 + \upsilon^{-1} \mathbf{c}_2 \otimes \mathbf{c}_2 + \mathbf{c}_3 \otimes \mathbf{c}_3, \quad 0 < \upsilon < \infty \tag{92}$$

The nominal stress is a corresponding "*shear*", in fact a triaxial state of stress π_1, π_2, π_3 defined by

$$\begin{aligned} \mathbf{P}(n) &= \pi_1(\upsilon; n)\mathbf{c}_1 \otimes \mathbf{c}_1 + \pi_2(\upsilon; n)\mathbf{c}_2 \otimes \mathbf{c}_2 + \pi_3(\upsilon; n)\mathbf{c}_3 \otimes \mathbf{c}_3 \\ \pi_1(\upsilon; n) &= \left\{\lambda[e_n(\upsilon^2) + e_n(\upsilon^{-2})] + 2\mu e_n(\upsilon^2)\right\} 2\upsilon e_n'(\upsilon^2) \equiv \pi(\upsilon; n) \\ \pi_2(\upsilon; n) &= \left\{\lambda[e_n(\upsilon^{-2}) + e_n(\upsilon^2)] + 2\mu e_n(\upsilon^{-2})\right\} 2\upsilon^{-1} e_n'(\upsilon^{-2}) = \pi(\upsilon^{-1}; n) \\ \pi_3(\upsilon; n) &= \lambda[e_n(\upsilon^2) + e_n(\upsilon^{-2})] \end{aligned} \tag{93}$$

The matrices of **F** and **P** in the orthonormal spectral basis are

$$[\mathbf{F}] = \begin{bmatrix} \upsilon & 0 & 0 \\ 0 & \upsilon^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [\mathbf{P}(n)] = \begin{bmatrix} \pi(\upsilon; n) & 0 & 0 \\ 0 & \pi(\upsilon^{-1}; n) & 0 \\ 0 & 0 & \pi_3(\upsilon; n) \end{bmatrix}$$

The nominal shear stress π can be expressed in terms of a *metric nominal glide scale function* g_n which depends on ν besides υ

$$\begin{aligned} \pi(\upsilon; n) &\equiv 2\mu g_n(\upsilon; \nu), \quad \pi(1; n) = 0 \\ g_n(\upsilon; \nu) &= 2\upsilon e_n'(\upsilon^2)\left[\frac{1-\nu}{1-2\nu} e_n(\upsilon^2) + \frac{\nu}{1-2\nu} e_n(\upsilon^{-2})\right], \quad g_n(1; \nu) = 0 \\ &= \frac{1}{1-2\nu}\left[\frac{2+n}{4}\frac{2(1-2\nu)+n}{8}\upsilon^3 - \frac{n}{4}\frac{2+n}{4}\upsilon + \frac{n\nu}{4}\upsilon^{-1} - \frac{n}{4}\frac{2-n}{4}\upsilon^{-3} - \frac{2-n}{4}\frac{2(1-2\nu)-n}{8}\upsilon^{-5}\right] \end{aligned} \tag{94}$$

Two remarkable properties of the metric and quasilogarithmic scales (unrevealed up to now) are the inversion-opposition and division–subtraction conversion rules (inherited from the logarithm)

$$e_n(\upsilon^{-2}) = -e_{-n}(\upsilon^2); \quad e_0(\upsilon^{-2}) = -e_0(\upsilon^2)$$
$$g_n(\upsilon;0) = f_n(\upsilon); \quad g_0(\upsilon;\nu) = f_0(\upsilon) \tag{95}$$

Studying the sign of the derivative $g'_n(\upsilon;\nu)$, it can be shown that the metric law is everywhere *monotone* in a pure glide only if

$$-1.703\min[1,1-2\nu] < n < 1.979\min[1,1-2\nu]$$
$$\forall \nu \in [-1,1/2] \ \ \forall \upsilon \in (0,\infty) \tag{96}$$

In particular, it is monotone for $n = 0$ ($\forall \nu$) or less useful for $-1 \leqslant \nu \leqslant 0$ ($\forall n, -1.703 < n < 1.979$). For the more useful range $1/10 \leqslant \nu \leqslant 1/2$, it remains everywhere monotone only for $-2(1-2\nu) \leqslant n \leqslant 2(1-2\nu)$. Conversely, the 3D υ-region around unity over which it remains monotone decreases as ν increases, to shrink into a narrow channel along the trissectrix $\upsilon_1 = \upsilon_2 = \upsilon_3$ when the material becomes incompressible. A similar trend was observed by Hill with the stretch family (Hill, 1968). In short, the quasilogarithmic stress–strain pair $n = 0$ is preferable for (nearly) incompressible materials (rubber elasticity, metal elastoplasticity).

The graphs $g = g_n(\upsilon;\nu)$ of the nominal glide scale function (and hence those $\tau = \tau(\upsilon; n)$ of shear–glide laws for $2\mu = 1$) are plotted in Fig. 6 for $\nu = 0, 1/3, 1/2$ and the integer and critical values of n.

A pure glide is difficult to realize. A pure shear is easier to obtain.

6.7. Pure shear

Consider next a "*glide*", in fact a triaxial stretch, given , υ_1, υ_2, υ_3 ($0 < \upsilon_c < \infty$)

$$\mathbf{F} = \upsilon_1 \mathbf{c}_1 \otimes \mathbf{c}_1 + \upsilon_2 \mathbf{c}_2 \otimes \mathbf{c}_2 + \upsilon_3 \mathbf{c}_3 \otimes \mathbf{c}_3 \tag{97}$$

such that the corresponding nominal stress is a *pure shear*, i.e. a *plane stress* with *opposite* principal stresses, characterized by $\pi_1 = -\pi_2 = \pi$; $\pi_3 = 0$

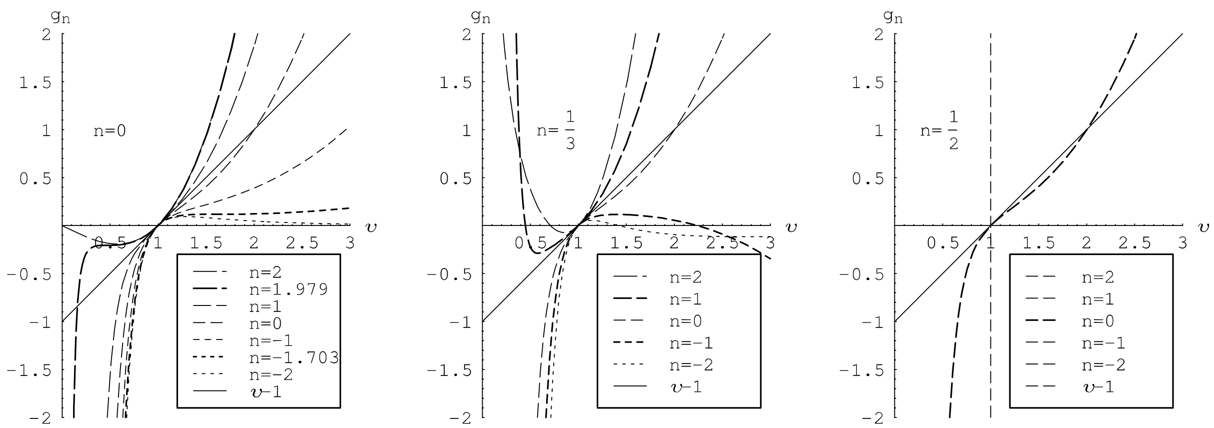

Fig. 6. Graphs $g = g_n(\upsilon)$ of the nominal glide scale function g_n (and hereby of the shear–glide laws), for $\nu = 0, 1/3, 1/2$ and the relevant values of n.

$$\mathbf{P}(n) = \pi(\upsilon_c; n)\mathbf{c}_1 \otimes \mathbf{c}_1 - \pi(\upsilon_c; n)\mathbf{c}_2 \otimes \mathbf{c}_2$$
$$\pi(\upsilon_c; n) \equiv \{\lambda[e_n(\upsilon_1^2) + e_n(\upsilon_2^2) + e_n(\upsilon_3^2)] + 2\mu e_n(\upsilon_1^2)\}2\upsilon_1 e_n'(\upsilon_1^2)$$
$$= -\{\lambda[e_n(\upsilon_1^2) + e_n(\upsilon_2^2) + e_n(\upsilon_3^2)] + 2\mu e_n(\upsilon_2^2)\}2\upsilon_2 e_n'(\upsilon_2^2)$$
$$\pi_3(\upsilon_c; n) = \{\lambda[e_n(\upsilon_1^2) + e_n(\upsilon_2^2) + e_n(\upsilon_3^2)] + 2\mu e_n(\upsilon_3^2)\}2\upsilon_3 e_n'(\upsilon_3^2) = 0 \quad (98)$$

The condition of zero normal stress implies that

$$e_n(\upsilon_3^2) = -\frac{\lambda}{\lambda + 2\mu}[e_n(\upsilon_1^2) + e_n(\upsilon_2^2)] = -\frac{v}{1-v}[e_n(\upsilon_1^2) + e_n(\upsilon_2^2)]$$
$$\Longleftrightarrow \upsilon_3(\upsilon_1, \upsilon_2) = \sqrt{e_n^{-1}\left\{-\frac{v}{1-v}[e_n(\upsilon_1^2) + e_n(\upsilon_2^2)]\right\}} \quad (99)$$

It follows that the plane stresses are equal and opposite to

$$\pi(\upsilon_1, \upsilon_2; n) = \frac{\varepsilon}{1-v^2}[e_n(\upsilon_1^2) + v e_n(\upsilon_2^2)]2\upsilon_1 e_n'(\upsilon_1^2)$$
$$= -\frac{\varepsilon}{1-v^2}[e_n(\upsilon_2^2) + v e_n(\upsilon_1^2)]2\upsilon_2 e_n'(\upsilon_2^2)$$
$$\Longleftrightarrow [e_n(\upsilon_1^2) + v e_n(\upsilon_2^2)]2\upsilon_1 e_n'(\upsilon_1^2) = -[e_n(\upsilon_2^2) + v e_n(\upsilon_1^2)]2\upsilon_2 e_n'(\upsilon_2^2) \quad (100)$$

In principle, the opposite stresses condition (100) can be used to express υ_2 in terms of $\upsilon_1 \equiv \upsilon$ and hereby to find the "glide": $\upsilon; \upsilon_2(\upsilon); \upsilon_3[\upsilon, \upsilon_2(\upsilon)]$, but a closed form expression is complicated.

The matrices of **F** and **P** in the orthonormal spectral basis are

$$[\mathbf{F}] = \begin{bmatrix} \upsilon & 0 & 0 \\ 0 & \upsilon_2(\upsilon) & 0 \\ 0 & 0 & \upsilon_3(\upsilon) \end{bmatrix}, \quad [\mathbf{P}(n)] = \begin{bmatrix} \pi(\upsilon; n) & 0 & 0 \\ 0 & -\pi(\upsilon; n) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

6.8. Summary

Application of the metric nominal law (60) to homogeneous stress–strain states has clarified its properties. It was shown that the law is *monotone* in the following specific ranges of n

Homogeneous stress–strain	Metric law monotony range of n for $-1 \leqslant v \leqslant 1/2$ and for $0 < \upsilon < \infty$
Tension–dilatation	$-1.703 < n < 1.979$
Pressure–volume	$-0.970 < n \leqslant 2$
Traction–elongation	$-1.703 < n < 1.979$
Shear–glide	$-1.703\min[1, 1-2v] < n < 1.979\min[1, 1-2v]$

Therefore, within the range $-0.970\min[1, 1-2v] < n < 1.979\min[1, 1-2v]$ for all $-1 \leqslant v \leqslant 1/2$ and more practically within $-(1-2v) \leqslant n \leqslant 2(1-2v)$ for all $2/100 \leqslant v \leqslant 1/2$, the metric law is much better behaved than the extreme laws obtained for $n = 2$ (the classical StVenant–Kirchhoff law) and $n = -2$, which are not monotone in compression and in tension, respectively. The quasilogarithmic law $n = 0$ remains monotone in pure shear–glide even for incompressible materials $v = 1/2$.

7. Conclusion

Introducing the progressive definitions of generalized, isotropic and simple strains, a family of metric strains was proposed that represents a second-order approximation of the Seth–Hill stretch family, but remains easier to calculate. The corresponding metric strain rates were then expressed in terms of the nominal strains and their rates. Using the previous definitions of strains, a family of conjugate metric stresses was derived, firstly by a static analysis and secondly by energetic duality.

The proposed metric strain–stress pairs were then used to formulate linear elastic metric laws that generate non-linear nominal laws in large transformations. The metric and nominal versions of the strain energy density, stress and stiffness tensors were calculated. In the case of material isotropy, the corresponding spectral energy density was also calculated.

The definition of the rank-one convexity condition of the nominal strain energy, necessary for existence of a solution to the equilibrium equations with metric laws, was recalled. Using the hypothesis of material isotropy, rank-one convexity was translated in conditions on the spectral energy density. The regions of the principal stretch space where the rank-one convexity condition is satisfied were delimited numerically and to a lower extent analytically. No metric law was found to be rank-one convex everywhere, but the quasilinear ($n = 1$) and quasilogarithmic ($n = 0$) laws proved to be so over substantially larger areas around the original state than the Green–Kirchhoff ($n = 2$) or the Karni–Rivlin ($n = -2$) ones.

Finally, the monotony of the metric nominal stress–strain graphs corresponding to the classical rheological experiments of dilatation, simple elongation, simple traction and pure glide was analyzed. The quasilinear option exhibited the minimal geometric nonlinearity and showed an inflexion point at the origin in dilatation, elongation and traction. The metric laws were not monotonic in pure glide when $\nu = 1/2$, except for the quasilogarithmic one.

To conclude, the quasilinear ($n = 1$) and quasilogarithmic ($n = 0$) strain–stress pairs can be used to extend the application of most existing small strain an-isotropic rheological laws, such as linear elasticity, conewise linear elasticity, linear viscoelasticity, elastoplasticity, damage,... to substantially larger strains than the Green–Kirchhoff (or Karni–Rivlin) pairs.

Acknowledgement

The authors are grateful to Bernard Dacorogna for several instructive discussions concerning the characterizations of the isotropic convexity variants, specially rank-one convexity.

References

Atluri, S., 1984. Alternate stress and conjugate strain measures, and mixed variational formulations involving rigid rotations, for computational analyses of finitely deformed solids, with applications to plates and shells, I. Comput. Struct. 18 (1), 93–116.

Ball, J., 1977a. Convexity conditions and existence theorems in nonlinear elasticity. Arch. Ratl. Mech. Anal. 63, 337–403.

Ball, J., 1977b. Constitutive inequalities and existence theorems in nonlinear elasticity. In: Knops, R. (Ed.), Nonlinear Analysis and Mechanics. Pitman, pp. 187–241.

Ball, J., 1984. Differentiability properties of symmetric and isotropic functions. Duke Math. J. 51, 699–728.

Billington, E., 1985. Referential stress tensor. Acta Mech. 55, 263–266.

Billington, E., 1986. The Poynting effect. Acta Mech. 58, 19–31.

Bruhns, O.T., Xiao, H., Meyers, A., 2001. Constitutive inequalities for an isotropic elastic strain-energy function based on Hencky's logarithmic strain tensor. Proc. R. Soc. Lond. A 457, 2207–2226.

Carlson, D., Hoger, A., 1986. The derivative of a tensor valued function of a tensor. Quart. Appl. Math. 44, 409–423.

Chadwick, P., 1976. Continuum Mechanics. Concise Theory and Problems. Allen and Unwin.

Ciarlet, P., 1988. Mathmatical Elasticity, Three-Dimensional Elasticity, vol. 1. North-Holland.

Curnier, A., 1994. Computational Methods in Solid Mechanics. Kluwer Academic Publishers.
Curnier, A., 2004. Mécanique des solides déformables. Cinématique, Dynamique, Energétique. Presses Polytechniques Universitaires Romandes, Lausanne.
Curnier, A., Rakotomanana, L., 1984. Generalized strain and stress measures, Document, LMA, DGM, EPFL, Lausanne.
Curnier, A., Rakotomanana, L., 1991. Generalized strain and stress measures: critical survey and new results. Eng. Trans. 39 (3–4), 461–538.
Dacorogna, B., 1988. Direct Methods in the Calculus of Variations. Springer-Verlag.
Dacorogna, B., 2001. Necessary and sufficient conditions for strong ellipticity of isotropic functions in any dimension. Discr. Continuous Dyn. Syst.—Ser. B 1 (2), 257–263.
Doyle, T., Ericksen, J., 1956. Nonlinear elasticity, Adv. Appl. Mech., vol. 4. Academic Press, pp. 53–115.
Eringen, C., 1975. Continuum Physics II—Continuum Mechanics of Single-Substance Bodies. Academic Press.
Fitzgerald, J., 1980. A tensorial Hencky measure of strain and strain rate for finite deformation. Dev. Theo. Appl. Mech. 10, 635–648.
Green, E., Rivlin, R., 1964. On Cauchy's equations of motion. J. Appl. Math. Phys. 15, 290–292.
Guan-Suo, D., Qing-Wen, R., Zhu-Jiang, S., 1999. Time rates of Hill's strain tensors. J. Elasticity 54, 129–140.
Guo, Z., 1984. Rates of stretch tensors. J. Elasticity 14, 263–267.
Guo, Z., Dubey, R., 1984. Basic aspects of Hill's method in solid mechanics. SM Arch. Nijhoff, 353–380.
Gurtin, M., 1981. An Introduction to Continuum Mechanics. Academic Press.
Hadeler, K., 1983. On copositive matrices. Linear Algebra Appl. 49, 79–89.
Hill, R., 1968. On constitutive inequalities for simple materials, I and II. J. Mech. Phys. Solids 16, 229–242, 315–322.
Hill, R., 1978. Aspects of invariance in solid mechanics. Adv. Appl. Mech., vol. 18. Academic Press, pp. 1–75.
Hoger, A., Carlson, D., 1984a. Determination of the stretch and rotation in the polar decomposition of the deformation gradient. Quart. Appl. Math., 113–117.
Hoger, A., Carlson, D., 1984b. On the derivative of the square root of a tensor and Guo's rate theorem. J. Elasticity 14, 329–336.
Karni, Z., Reiner, M., 1964. The general measure of deformation. In: Second-order Effects in Elasticity Plasticity and Fluid Dynamics. Macmillan, pp. 217–227.
Leblond, J., 1992. A constitutive inequality for hyperelastic materials in finite strain. Eur. J. Mech. A/Solids 11 (4), 447–466.
MacVean, D., 1968. Die Elementararbeit in einem Kontinuum und die Zuordnung von Spannungs und Verzerrungstensoren. Zeit. Ang. Math. Phys. 19, 157–185.
Man, C.-S., Guo, Z.-H., 1993. A basis-free formula for time rates of Hill's strain tensors. Int. J. Solids Struct. 30, 2819–2842.
Marsden, J., Hughes, T., 1983. Mathematical Foundation of Elasticity. Prentice Hall.
Moreau, J.-J., 1979. Lois de l'élasticité en grandes déformations. Séminaire d'analyse convexe-12, Montpellier.
Morrey, C., 1952. Quasiconvexity and the semicontinuity of multiple integrals. Pacific J. Math. 2, 25–53.
Ogden, R., 1974. On stress rates in solid mechanics with applications to elasticity theory. Proc. Cambridge Phil. Soc. 75, 303–319.
Ogden, R., 1984. Non-linear Elastic Deformation. Ellis Horwood.
Pietrzak, G., 1997. Continuum mechanics modelling and augmented Lagrangian formulation of large deformation frictional contact problems. Doctorate thesis, LMA, DGM, EPFL, Lausanne.
Pietrzak, G., Curnier, A., 1999. Large deformation frictional contact mechanics: continuum formulation and augmented Lagrangian treatment. Comput. Meth. Appl. Mech. Eng. 177, 351–381.
Rosakis, P., Simpson, H., 1995. On the relation between polyconvexity and rank-one convexity in nonlinear elasticity. J. Elasticity 37, 113–137.
Rosati, L., 2000. Derivatives and rates of the stretch and rotation. J. Elasticity 56, 213–230.
Scheidler, M., 1991. Time rates of generalized strain tensors. Part I: Component formulas. Part II: Approximate basis-free formulas. Mech. Mater. 11, 199–219.
Seth, B., 1964. Generalized strain measure with applications to physical problems. In: Second-order Effects in Elasticity, Plasticity and Fluid Dynamics. Macmillan, pp. 162–172.
Silhavy, M., 1997. The Mechanics and Thermodynamics of Continuous Media. Springer-Verlag.
Silhavy, M., 1999. On isotropic rank 1 convex functions. Proc. Roy. Soc. Edinburgh 129A, 1081–1105.
Simpson, H., Spector, S., 1983. On copositive matrices and strong ellipticity for isotropic elastic materials. Arch. Ratl. Mech. Anal. 84 (1), 55–68.
Truesdell, C., 1952. The mechanical foundations of elasticity and fluid dynamics. J. Ratl. Mech. Anal. 1, 125–300.
Truesdell, C., Noll, W., 1965. The Nonlinear Field Theories of Mechanics, second ed. Springer-Verlag, 1992.
Truesdell, C., Toupin, R., 1960. The Classical Field Theories, second ed. Encyclopedia of Physics, vol. III/1. Springer-Verlag.
Xiao, H., 1995. Unified explicit basis-free expressions for time rate and conjugate stress of an arbitrary Hill's strain. Int. J. Solids Struct. 32, 437–444.

Xiao, H., Bruhns, O., Meyers, A., 1998a. Objective corotational rates and unified work-conjugacy relations between Eulerian and Lagrangean strain and stress measures. Arch. Mech. 50, 1015–1045.
Xiao, H., Bruhns, O., Meyers, A., 1998b. Strain rates and material spins. J. Elasticity 52, 1–41.
Ziegler, H., MacVean, D., 1967. On the notion of an elastic solid. Recent Progress in Appl. Mech, The Folke Odqvist 561–572.

Further reading

Rajaonah, R., 1990. Etude d'une loi de comportement viscoélastique. Doctorate thesis, LMA, DGM, EPFL, Lausanne.
Rakotomanana, L., 1986. Analyse théorique et numérique des grandes déformations en Mécanique des milieux continus. Doctorate thesis, LMA, DGM, EPFL, Lausanne.
Reiner, M., 1956. On isotropic functions and the measure of deformation. J. Appl. Math. Phys. 7, 377–393.
Simo, J., Hughes, T., 1986. Computational Inelasticity. Springer-Verlag.